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On Oscillation of Nonlinear Differential Equations of Second Order

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Abstract

In this paper, we present some new sufficient conditions for the oscillation of all solutions of the second order non-linear ordinary differential equation of the form

$$\left(r(t)\dot{x}(t)\right)^{\bullet} + q(t)\Phi(g(x(t)), r(t)\dot{x}(t)) = H(t, x(t))$$

where q and r are continuous functions on the interval $[t_0, \infty)$, $t_0 \geq 0$, $r(t)$ is a positive function, g is continuously differentiable function on the real line R except possibly at 0 with $xg(x) > 0$ and $g'(x) \geq k > 0$ for all $x \neq 0$, Φ is a continuous function on RxR with $u\Phi(u,v) > 0$ for all $u \neq 0$ and $\Phi(\lambda u, \lambda v) = \lambda\Phi(u,v)$ for any $\lambda \in (0,\infty)$ and H is a continuous function on $[t_0, \infty)\times$R with $H(t, x(t))/g(x(t)) \leq p(t)$ for all $x \neq 0$ and $t \geq t_0$. The oscillatory behavior of ordinary differential equations has been extensively studied by many authors, see for examples [1-14] and the references therein. This research work which is obtained using Riccati Technique, extends and improves many of the known results of oscillation in the literatures such as our oscillation results extend result of Wong and Yeh[14], result of Philos[9], result of Onose[8], result of Philos and Purnaras[10], result of E. M. Elabbasy[3], results of Greaf, Rankin and Spikes[5], results of Grace and Lalli[4] and results of Moussadek Rmail[7] and some other previous results. We illustrate our oscillation results and the improvement over other known oscillation conditions by examples, numerically are solved in MATLAB.

Keywords: *Oscillation conditions , numerically , ordinary differential equation , MATLAB*

1. Introduction

In this paper, we are concerned with the problem of oscillation of second order non-linear ordinary differential equation of the form

$$\left(r(t)\dot{x}(t)\right)^{\bullet} + q(t)\Phi\left(g(x(t)), r(t)\dot{x}(t)\right) = H\left(t, x(t)\right) \qquad (E)$$

where q and r are continuous functions on the interval $[t_0, \infty)$, $t_0 \geq 0$, $r(t)$ is a positive function, g is continuously differentiable function on the real line $\mathbb{R}$ except possibly at 0 with $xg(x) > 0$ and $g'(x) \geq k > 0$ for all $x \neq 0$, Φ is a continuous function on $\mathbb{R}x\mathbb{R}$ with $u\Phi(u,v) > 0$ for all $u \neq 0$ and $\Phi(\lambda u, \lambda v) = \lambda\Phi(u,v)$ for any $\lambda \in (0,\infty)$ and H is a continuous function on $[t_0, \infty) \times \mathbb{R}$ with $H(t, x(t))/g(x(t)) \leq p(t)$ for all $x \neq 0$ and $t \geq t_0$. A solution $x(t)$ of the differential equation (E) is said to be oscillatory if it has arbitrary large zeros, and otherwise it is called non-oscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory, and otherwise it is called non-oscillatory.

Equation (E) is said to be superlinear if

$$0 < \int_{\pm\varepsilon}^{\pm\infty} \frac{du}{g(u)} < \infty \text{ for every all } \varepsilon > 0.$$

The oscillatory behavior of ordinary differential equations has been extensively studied by many authors, see for examples [1-14] and the references therein.

Onose [8] studied the equation

$$\ddot{x}(t) + q(t)g(x(t)) = 0 \qquad (1)$$

and proved that if

(1) $\liminf\limits_{t\to\infty} \int_{t_0}^{t} q(s)\, ds \geq 0,$

(2) $\limsup\limits_{t\to\infty} \int_{t_0}^{t} q(s) ds = \infty,$

then the equation (1) is oscillatory. Philos [9] and Wong and Yeh [14] considered the equation (1) and gave extensions to the result of Onose [8]. Philos and Purnaras [10] have studied the equation (1) and supposed that the superlinear differential equation (1) is oscillatory if

(1) $\liminf_{t\to\infty} \frac{1}{t^{n-1}} \int_{t_0}^{t} (t-s)^{n-1} q(s)\, ds > -\infty$ for some integer $n \geq 1$,

(2) $\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \left(\int_{t_0}^{s} q(u)\, du \right)^2 ds = \infty.$

E. M. Elabbasy [3] has studied the equation

$$\left(r(t)\, \dot{x}(t) \right)^{\bullet} + q(t)\; g(x(t)) = 0 \qquad (2)$$

and showed that the equation (2) is oscillatory if

(1) $\liminf_{t\to\infty} \int_{t_0}^{t} \rho(s) q(s)\, ds > -\infty,$

(2) $\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \left(\int_{t_0}^{s} \rho(u) q(u)\, du \right)^2 ds = \infty.$

2. Main Results

In this section, we state and prove our main results as follow:

Theorem 2.1

Suppose that

(1) $\frac{1}{\Phi(1,u)} < \frac{1}{C_0}, C_0 > 0,$

(2) $q(t) > 0 \; for\, all\, t \geq t_0,$

and there exists continuously differentiable function $\rho:[t_0,\infty)\to(0,\infty)$ such that $\dot{\rho}(t)\geq 0 \;\; on \; [t_0,\infty)$ and

(3) $\displaystyle\int_{t_0}^{\infty}\frac{ds}{\rho(s)r(s)}=\infty,$

(4) $\displaystyle\int_{t_0}^{\infty}\rho(s)[C_0 q(s)-p(s)]ds<\infty,$

(5) $\displaystyle\liminf_{t\to\infty}\left[\int_{t_0}^{\infty}\Psi(s)\,ds\right]\geq 0$ for all large t,

(6) $\displaystyle\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{\rho(s)r(s)}\int_{s}^{\infty}\Psi(u)du ds=\infty.$

Then, every solution of superlinear equation (*E*) is oscillatory.

Proof: Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (*E*) such that $x(t)>0 \;\; on \; [T,\infty) \; for some \; T\geq t_0\geq 0.$ Define

$$\omega(t)=\frac{r(t)\dot{x}(t)}{g(x(t))},\; t\geq T$$

This and (*E*) imply

$$\dot{\omega}(t)=\left(\frac{r(t)\dot{x}(t)}{g(x(t))}\right)^{\bullet}\leq p(t)-q(t)\Phi(1,\omega(t))-\frac{k\,r(t)\dot{x}^2(t)}{g^2(x(t))},t\geq T \qquad (2-1)$$

From condition (1), for $t\geq T$, we have

$$\left(\frac{r(t)\dot{x}(t)}{g(x(t))}\right)^{\bullet}\leq -[C_0 q(t)-p(t)]-\frac{k\,r(t)\dot{x}^2(t)}{g^2(x(t))},\; t\geq T$$

We multiply the last inequality by $\rho(t)$ and integrate form T to t, we have

$$\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \le \frac{\rho(T)r(T)\dot{x}(T)}{g(x(T))} - \int_T^t \rho(s)\big(C_0 q(s) - p(s)\big)\, ds + \int_T^t \left[\dot{\rho}(s)\omega(s) - k\frac{\rho(s)}{r(s)}\omega^2(s)\right] ds$$

Let $C_1 = \dfrac{\rho(T)r(T)\dot{x}(T)}{g(x(T))}$ *and* $\eta(t) = \omega(t) - \dfrac{\dot{\rho}(t)r(t)}{2k\rho(t)}$.

$$\begin{aligned}\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} &\le C_1 - \int_T^t \rho(s)\big(C_0 q(s) - p(s)\big)\, ds + \int_T^t k\frac{\rho(s)}{r(s)}\left(\eta^2(s) - \left(\frac{\dot{\rho}(s)r(s)}{2k\rho(s)}\right)^2\right) ds \\ &\le C_1 - \int_T^t \left[\rho(s)\big(C_0 q(s) - p(s)\big) - \frac{\dot{\rho}^2(s)r(s)}{4k\rho(s)}\right] ds \\ &\le C_1 - \int_T^t \Psi(s)ds \qquad (2-2)\end{aligned}$$

From inequality (2-2), we have

$$\int_T^t \Psi(s)ds \le \frac{\rho(T)r(T)\dot{x}(T)}{g(x(T))} - \frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))}$$

Now, we consider three cases for $\dot{x}(t)$

Case 1: If $\dot{x}(t) > 0$ $for\, t \ge T_1 \ge T$, then we get

$$\int_{T_1}^t \Psi(s)ds \le \frac{\rho(T_1)r(T_1)\dot{x}(T_1)}{g(x(T_1))} - \frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))}$$

Thus, for all $t \ge T_1$, we obtain

$$\int_t^\infty \Psi(s)ds \le \frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))}$$

We divide the last inequality $\rho(t)\,r(t)$ and integrate from T_1 to t, we obtain

$$\int_{T_1}^{t}\frac{1}{\rho(s)r(s)}\int_{s}^{\infty}\Psi(u)du ds \le \int_{T_1}^{t}\frac{\dot{x}(s)}{g(x(s))}ds$$

Since the equation (E) is superlinear, we get

$$\int_{T_1}^{t}\frac{1}{\rho(s)r(s)}\int_{s}^{\infty}\Psi(u)du ds \le \int_{T_1}^{t}\frac{\dot{x}(s)}{g(x(s))}ds = \int_{x(T_1)}^{x(t)}\frac{du}{g(u)} < \infty,$$

This contradicts condition (6).

Case 2: If $\dot{x}(t)$ is oscillatory, then there exists a sequence τ_n in $[T,\infty)$ such that $\dot{x}(\tau_n) = 0$. Choose N large enough so that (5) holds. Then from inequality (2-2), we have

$$\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \le C_{\tau_n} - \int_{\tau_n}^{t}\Psi(s)ds$$

So

$$\limsup_{t\to\infty}\frac{\rho(t)r(t)\dot{x}(t)}{g(x(t))} \le C_{\tau_n} + \limsup_{t\to\infty}\left\{-\int_{\tau_n}^{t}\Psi(s)ds\right\} = C_{\tau_n} - \liminf_{t\to\infty}\left\{\int_{\tau_n}^{t}\Psi(s)ds\right\} < 0,$$

which contradicts the fact that $\dot{x}(t)$ oscillates.

Case 3: If $\dot{x}(t) < 0$ for $t \ge T_2 \ge T$. The condition (5) implies that there exists $T_3 \ge T_2$ such that

$$\int_{T_3}^{t}\left(C_0 q(s) - p(s)\right) ds \ge 0 \qquad for\ t \ge T_3$$

Multiplying the equation (E) by $\rho(t)$ and from the condition (1), for $t \ge T_3$ we have

$$\rho(t)\left(r(t)\dot{x}(t)\right)^{\bullet}+C_0\rho(t)g(x(t))q(t)\le \rho(t)g(x(t))p(t),\ t\ge T_3$$

Integrate the last inequality from T_3 to t, we obtain

$$\rho(t)r(t)\dot{x}(t)\le \rho(T_3)r(T_3)\dot{x}(T_3)+\int_{T_3}^{t}\dot{\rho}(s)r(s)\dot{x}(s)ds-g(x(t))\int_{T_3}^{t}\rho(s)\left(C_0q(s)-p(s)\right)ds$$

$$+\int_{T_3}^{t}g'(x(s))\dot{x}(s)\int_{T_3}^{s}\rho(u)\left(C_0q(u)-p(u)\right)du\,ds\ ,t\ge T_3$$

$$\le \rho(T_3)r(T_3)\dot{x}(T_3)-g(x(t))\int_{T_3}^{t}\rho(s)\left(C_0q(s)-p(s)\right)ds$$

$$+\int_{T_3}^{t}g'(x(s))\dot{x}(s)\int_{T_3}^{s}\rho(u)\left(C_0q(u)-p(u)\right)du\,ds,\ t\ge T_3$$

$$\le \rho(T_3)r(T_3)\dot{x}(T_3),\ t\ge T_3.$$

Dividing the last inequality by $\rho(t)r(t)$ and integrate from T_3 to t we obtain

$$x(t)\le x(T_3)+\rho(T_3)r(T_3)\dot{x}(T_3)\int_{T_3}^{t}\frac{ds}{\rho(s)r(s)}\to -\infty, as\, t\to\infty$$

which is a contradiction to the fact that $x(t)>0$ for $t\ge T$. Hence the proof is completed.

Example 2.1

Consider the following differential equation

$$\left(\dot{x}(t)/t\right)^{\bullet}+\frac{1}{t^3}\frac{x^{35}(t)}{2x^{28}(t)+\left(\dot{x}(t)/t\right)^4}=-\frac{x^{11}(t)}{x^4(t)+1},\ t>0.$$

Here $r(t)=1/t$, $q(t)=1/t^3$, $g(x)=x^7$, $\Phi(u,v)=u^5/\left(2u^4+v^4\right)$ and

$\dfrac{H(t,x(t))}{g(x(t))}=-\dfrac{x^4(t)}{x^4(t)+1}\le 0=p(t)$ *for all* $x\ne 0$ *and* $t>0$. Taking $\rho(t)=4>0$ such that

(1) $\int_{t_0}^{\infty} \rho(s)\left(C_0 q(s) - p(s)\right)ds = \frac{2C_0}{t_0^2} < \infty,$

(2) $\liminf_{t\to\infty}\left\{\int_{t_0}^{t}\Psi(s)ds\right\} = \liminf_{t\to\infty}\left\{\int_{t_0}^{t}\left[\rho(s)\left(C_0 q(s) - p(s)\right) - \frac{\rho^{\bullet 2}(s)r(s)}{4k\rho(s)}\right]ds\right\} = \frac{2C}{t_0^{\ 2}} > 0,$

(3) $\lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{\rho(s)r(s)}\int_{s}^{\infty}\Psi(u)\,duds = \lim_{t\to\infty}\int_{t_0}^{t}\frac{1}{\rho(s)r(s)}\int_{s}^{\infty}\left[\left(C_0 q(u) - p(u)\right) - \frac{\rho^{\bullet 2}(u)r(u)}{4k\rho(u)}\right]duds = \infty.$

All conditions of theorem 2.1 are satisfied and hence every solution of the given equation is oscillatory. To ensure that our result in theorem 2.1 is true we also find the numerical solutions of the given differential equation in example2.1using the Runge Kutta method of fourth order. We have $\ddot{x}(t) = f(t, x(t), \dot{x}(t)) = -\frac{x^{11}}{x^4 + 1} - \frac{x^{35}}{2x^{28} + \dot{x}^4(t)}$

with initial conditions $x(1) = 1, \ \dot{x}(1) = 1$ on the chosen interval $[1,100]$ and finding values of the functions *r*, *q* and *f* where we consider $H(t, x) = f(t)\,l(x)$ at *t=1*, *n=500* and *h=0.198*

Table 1: Numerical solution of ODE1

K	t_k	$x(t_k)$
1	1	1
2	1.198	1.166
3	1.396	1.2043
.	.	.
.	.	.
10	2.782	-0.1516
11	2.98	-0.3605
12	3.178	-0.5694
.	.	.
.	.	.
23	5.356	0.1301
24	5.554	0.3383
25	5.752	0.5465
.	.	.
.	.	.
36	7.93	-0.1016
37	8.128	-0.3091
38	8.326	-0.5166

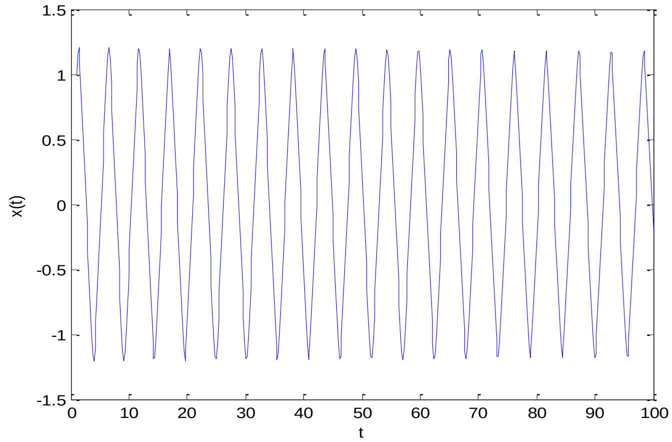

Figure1: Solution curve of ODE 1

As we see in Figure 1, all solutions of the given differential equation in Example 2.1 are oscillatory.

Remark 2.1: Theorem 2.1extends result of Wong and Yeh [14], result of Philos [9], result of Onose [8] and result of Philos and Purnaras [10]as $r(t) \equiv 1$, $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t)) \equiv 0$ and result of E. M. Elabbasy [3] as $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv g(x(t))$ and $H(t, x(t)) \equiv 0$. Also our theorem improves theorem 2 of Greaf, Rankin and Spikes [5], theorem 1 of Grace and Lalli [4] and theorem 2 of Moussadek Rmail [7].

Theorem 2.2: Suppose, in addition to the condition (2) holds that

(7) $\dot{r}(t) \le 0 \;\; for\, all\, t \ge t_0 \;\; and \;\; (r(t)q(t))^{\bullet} \ge 0 \;\; for\, all\, t \ge t_0.$

(8) $\Phi(1, v) \ge v \;\; for\, all\, v \ne 0.$

(9) $\limsup_{t\to\infty} \frac{1}{t}\int_T^t \left[A_2 r(s)q(s) - \int_{t_0}^s p(u)du \right] ds = \infty,$

where, $p:[t_0, \infty) \to (0, \infty)$, then every solution of superlinear equation (E) is oscillatory.

Proof: Without loss of generality, we may assume that there exists a solution $x(t)$ of equation (E) such that $x(t) > 0$ *on* $[T, \infty)$ *for some* $T \geq t_0 \geq 0$. From inequality (2-1) and the condition (8), we get

$$\left(\frac{r(t)\dot{x}(t)}{g(x(t))} \right)^{\bullet} \leq p(t) - q(t)\omega(t).$$

Integrate the last inequality from T to t, we obtain

$$\frac{r(t)\dot{x}(t)}{g(x(t))} \leq A_1 + \int_T^t p(s)ds - \int_T^t r(s)q(s)\frac{\dot{x}(s)}{g(x(s))}ds \qquad (2-3)$$

where $A_1 = \dfrac{r(T)\dot{x}(T)}{g(x(T))}$

By condition (7) and the Bonnet's theorem, we see that for each $t \geq T$, there exists $T_2 \in [T, t]$ such that

$$\int_T^t r(s)q(s)\frac{\dot{x}(s)}{g(x(s))}ds = r(t)q(t)\int_{T_2}^t \frac{\dot{x}(s)}{g(x(s))}ds = r(t)q(t)\int_{x(T_2)}^{x(t)} \frac{du}{g(u)}$$

Since $r(t)q(t) \geq 0$ and the equation (E) is superlinear we have

$$\int_{x(T_2)}^{x(t)} \frac{du}{g(u)} < \begin{cases} 0 & , If\ x(t) < x(T_2) \\ \displaystyle\int_{x(T_2)}^{\infty} \frac{du}{g(u)} & , If\ x(t) \geq x(T_2) \end{cases}$$

Thus, it follows that

$$\int_T^t r(s)q(s)\frac{\dot{x}(s)}{g(x(s))}ds \geq A_2 r(t)q(t) \text{ where } A_2 = \inf \int_{x(T_2)}^{x(t)} \frac{du}{g(u)}$$

Thus the inequality (2-3) becomes

$$\frac{r(t)\dot{x}(t)}{g(x(t))} \le A_1 + \int_T^t p(s)ds - A_2 r(t)q(t)$$

Integrate the last inequality from T to t, we have

$$\int_T^t \frac{r(s)\dot{x}(s)}{g(x(s))}ds \le A_1(t-T) - \int_T^t \left[A_2 r(s)q(s) - \int_T^s p(u)du \right] ds$$

Since $r(t)$ is positive and non-increasing for $t \ge T$, the equation (E) is superlinear and by Bonnet's theorem, there exists $\beta_t \in [t,T]$ such that

$$\int_T^t \frac{r(s)\dot{x}(s)}{g(x(s))}ds = r(T)\int_{x(T)}^{x(\beta_t)} \frac{du}{g(u)} \ge A_3 r(T), \text{ where } A_3 = \inf \int_{x(T)}^{x(\beta_t)} \frac{du}{g(u)}$$

Thus, for $t \ge T$, we have

$$\int_T^t \left[A_2 r(s)q(s) - \int_T^s p(u)du \right] ds \le A_1(t-T) - A_3 r(T)$$

Dividing the last inequality by t and taking the limit superior on both sides, we obtain

$$\limsup_{t\to\infty} \frac{1}{t}\int_T^t \left[A_2 r(s)q(s) - \int_T^s p(u)du \right] ds \le \limsup_{t\to\infty}\; A_1(1-\frac{T}{t}) - \limsup_{t\to\infty}\frac{A_3 r(T)}{t} < \infty,$$

as $t \to \infty$, which contradicts to the condition (9). Hence the proof is completed.

Example2.2: Consider the following differential equation

$$\left(\dot{x}(t)/t\right)^{\bullet} + t^4\left(x^5(t) + \frac{x^{15}(t)}{x^{10}(t) + \left(\dot{x}(t)/t\right)^2} \right) = \frac{x^7(t)\cos(x(t))}{t^4(x^2(t)+1)},\ t > 0.$$

Here, We have

$$r(t)=1/t,\, q(t)=t^4,\, g(x)=x^5,\ \Phi(u,v)=u+\frac{u^3}{u^2+v^2} \quad \text{and} \quad \frac{H(t,x(t))}{g(x(t))}=\frac{x^2(t)\cos(x(t))}{t^4(x^2(t)+1)}\le\frac{1}{t^4}=p(t)$$

for all $t>0$ *and* $x\neq 0$.

$$\limsup_{t\to\infty}\frac{1}{t}\int_T^t\left[A_2 r(s)q(s)-\int_T^s p(u)du\right]ds=\limsup_{t\to\infty}\frac{1}{t}\int_T^t\left[A_2 s^3-6\int_T^s\frac{du}{u^4}\right]ds=\infty.$$

All conditions of theorem 2.2 are satisfied and hence every solution of the given equation is oscillatory. The numerical solutions of the given equation using the Runge Kutta method of fourth order (RK4) is as follows:

We have

$$\ddot{x}(t)=f(t,x(t),\dot{x}(t))=\frac{x^7(t)\cos(x(t))}{x^2(t)+1}-\left(x^5(t)+x^{15}(t)\Big/\left(x^{10}(t)+\dot{x}^2(t)\right)\right)$$

with initial conditions $x(1)=1,\ \dot{x}(1)=1$ on the chosen interval $[1,100]$ and finding values of the functions *r, q* and *f* where we consider $H(t,x(t))=f(t)l(x)$ at *t=1, n=500* and *h=0.198.*

Table 2: Numerical solution of ODE2

K	t_k	$x(t_k)$
1	1	1
2	1.98	1.1587
3	1.396	1.1841
.	.	.
.	.	.
9	2.584	-0.046
10	2.782	-0.2729
11	2.98	-0.4996
.	.	.
.	.	.
22	5.158	0.2073
23	5.356	0.4336
24	5.554	0.6593
.	.	.
.	.	.
34	7.534	-0.139
35	7.732	-0.3651
36	7.93	-0.5908

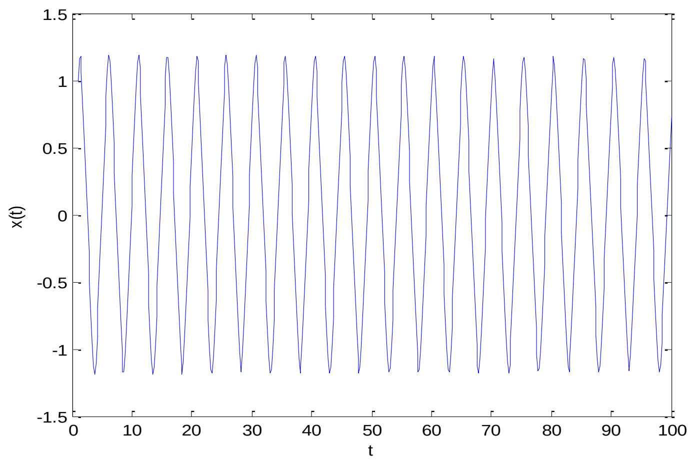

Figure2: Solution curve of ODE 2

As we see in Figure 2, all solutions of the given differential equation in Example 2.2 are oscillatory.

Remark2.2: If $(i)\, r(t) \equiv 1,\ (ii)\ \Phi(g(x(t)), r(t)\dot{x}(t)) \equiv \Phi(x(t), \dot{x}(t))\ and\, (iii)\ H(t, x(t)) \equiv 0,$ then theorem2.2 extends results of Bihari [1], Kartsatos[6]. In addition to (i) and (iii), if (ii) is $\Phi(g(x(t)), r(t)\dot{x}(t)) \equiv g(x(t))$ then theorem2.2 extends results of E. M. Elabbasy [3]. All results of Bihari [1], Kartsatos [6] and E. M. Elabbasy [3] can't be applied to the given equation in example2.2.

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