Advantages Of The Normal Derivative Method To Set Up The Integral Kernels For A Unit Disc

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Abstract

We implement the boundary element method for the Helmholtz equation for a two-dimensional unit disc with Dirichlet boundary conditions. This yields boundary integral equations of Fredholm kind. Here we discuss the advantages of the normal derivative method which leads to a second-kind Fredholm integral equation instead of dealing with the resulting first-kind Fredholm integral equation. This is shown by comparing the accuracy of the boundary functions which are computed using both types of integral equations. We point out that it may be advantageous for smooth boundary to use the normal derivative method to set up the boundary integral kernels which gets rid of the weakly-singular integrals..

Keywords: *First and Second Kinds Fredholm Integral Equations; Boundary Element Method; Green Function; Weakly Singular Integrals.*

1. Introduction

The Boundary Element Method (BEM) [1, 2, 3] is a powerful technique for obtaining an approximate solution for ordinary or partial differential equations that arise in scientific and engineering applications, such as wave scattering, radiation and propagation. The Boundary Integral Equations (BIEs) [4] are respectively derived for the wave-function or its normal derivative or for both quantities depending on whether Newmann, Dirichlet or Robin (mixed) boundary conditions are set on the boundary. So implementing the boundary element method leads to a Fredholm BIEs of first or second kinds [4]. The mechanism of the BEM is to discretize the boundary into a number of elements to compute the integrals numerically over such elements. The system of BIEs is then converted into a linear system of algebraic equations which can be solved numerically. The Fredholm equation of first and second kinds both possess a unique solution and well adopted for numerical calculations [2]. The BEM formulation is based on the Green second identity which requires that the involved functions to be twice continuously differentiable [1]. Non-smooth boundary with sharp corners and edges is commonly used in modelling engineering problems. However, for such boundaries one usually make the assumption that the boundary is piecewise twice differentiable, that is, the boundary is constructed as finite union of twice differentiable sub-intervals; nevertheless the whole boundary is not twice differentiable. Corner nodes cause problems for two reasons. Firstly, the normal vector at a corner is not well defined, and the normal derivatives change its value sharply across the corner. This leads to discontinuity across the corner. Secondly, at a corner node, the standard jump relation is no longer valid and needs to be adjusted to accommodate the corners as shown in [4]. Therefore, the treatment of corners in the BEM formulation requires great care in order to obtain an accurate numerical solution in the presence of corners. Yan and Lin [5] present a review article on the possible treatment of the corners problem.

In this paper, we consider a smooth boundary such as unit disc with Dirichlet Boundary Conditions (DBCs), these BCs are typical for plate or memebrance problems with clamped or fixed boundaries. The governing equation is the Helmholtz equation. This paper is structured as follows: in section 1 we set up the model and we show how to apply the BEM for the Helmholtz equation to construct the boundary functions and the quantization condition which can be used to compute the spectrum. In section 2, we discuss the advantages of the second kind Fredholm equation over the first kind Fredholm equation for the unit disc. Then we show a comparison of the accuracy of the boundary functions computed using both the first and second kind BIEs. In section 3, we show how to use the obtained boundary functions to compute the Green function at any interior point of the considered domain. A conclusion is drawn in section 4.

1. Derivation of the BIEs for the Helmholtz equation

The homogeneous Helmholtz equation is defined as

$$(\nabla_{\mathbf{q}}^2 + k^2)\psi(\mathbf{q}) = 0, \tag{1}$$

where ψ is the corresponding eigenfunction to the eigenvalue k, $\mathbf{q}$ is an interior point and ∇^2 is the two-dimensional Laplace operator in the Cartesian coordinates. We set the DBCs on the boundary of the domain D, that is

$$\psi(q) = 0, \quad \text{for } q \in \partial D\,.$$

The first step within the BEM formulations is to introduce the fundamental solution of the problem which is essential to establish the necessary BIEs. The fundamental solution of a differential equation is a solution with a unit point source equal to $\delta(\mathbf{q} - \mathbf{r})$ applied at a given, fixed source point $\mathbf{r}$. The following equation thus holds,

$$\left(\nabla_{\mathbf{q}}^2 + k^2\right)G_0(\mathbf{q}, \mathbf{r}; k) = -\delta(\mathbf{q} - \mathbf{r}), \tag{2}$$

where G_0 is the fundamental solution, also it is called the free-space Green function defined as,

$$G_0(\mathbf{q},\mathbf{r};k) = \frac{i}{4}H_0^{(1)}(k|\mathbf{q}-\mathbf{r}|). \tag{3}$$

The function $H_0^{(1)}(k|\mathbf{q}-\mathbf{r}|)$ denotes the Hankel function of the first kind and zeroth order [6], and $|\mathbf{q}-\mathbf{r}|$ is the distance between the source $\mathbf{r}$ and the observation point $\mathbf{q}$.

1.1 Derivation of the first kind Fredholm BIE for the Helmholtz equation

To derive the BIEs, one needs to multiply equation (1) and (2) by $G_0(\mathbf{q},\mathbf{r};k)$ and $\psi(\mathbf{q})$ respectively. Then subtract the two resulting equations, and integrate over the region *D* with an area element $dA_{\boldsymbol{q}}$, to obtain

$$\iint_D [G_0(\mathbf{q},\mathbf{r};k)\nabla_{\mathbf{q}}^2\psi(\mathbf{q}) - \psi(\mathbf{q})\nabla_{\mathbf{q}}^2 G_0(\mathbf{q},\mathbf{r};k)]dA_{\mathbf{q}} = \iint_D \delta(\mathbf{q}-\mathbf{r})\psi(\mathbf{q})dA_{\mathbf{q}}. \tag{4}$$

For the left hand side (LHS) of equation (4), one needs to make use of the Green second identity [1] and the integral on the right hand side (RHS) depends on the position of $\mathbf{r}$, and can be classified as the following,

$$\int_{\partial D}\left(G_0(q,\mathbf{r};k)\frac{\partial}{\partial n_q}\psi(q)\text{-}\psi(q)\frac{\partial}{\partial n_q}G_0(q,\mathbf{r};k)\right)dq = \begin{cases} \psi(\mathbf{r}), & \text{if } r \in D; \\ \frac{1}{2}\psi(\mathbf{r}), & \text{if } r \in \partial D; \\ 0, & \text{otherwise,} \end{cases} \tag{5}$$

where $\vec{n}_q$ denotes the outward unit normal vector at the boundary point q. The operator $\frac{\partial}{\partial n_q}$ denotes the directional derivative along the normal vector $\vec{n}_q$ at the boundary element q, that is,

$$\frac{\partial}{\partial n_q}G(q,\mathbf{r};k) = \vec{n}_q \cdot \nabla_q G(q,\mathbf{r};k).$$

The dot denotes the scalar product, ∇_q is the gradient operator with respect to q, and dq is the arc length element along the boundary ∂D. The direct substitution of DBCs, and letting $\boldsymbol{r} \to \beta \in \partial D$ in equation (5) leads to the following BIE,

$$\int_{\partial D} G_0(q,\beta;k)\frac{\partial}{\partial n_q}\psi(q)dq = 0. \tag{6}$$

This BIE is classified as Fredholm integral equation of the first kind, because the unknown function $\frac{\partial}{\partial n_q}\psi(q)$ appears only implicitly, that is under the integration sign. Note that, this BIE has a logarithmically divergent kernel (weakly-singular) at $q = \beta$. Actually, we can proceed with the BIE (6) after a careful treatment of the weak-singularity of $G_0(q, \beta; k)$. But a well-known trick [7, 8] to avoid this additional complication is to take the normal derivative of the BIE (5) with respect to **r** in the limit ($\mathbf{r} \rightarrow \beta \in \partial D$) transforming all G_0 terms to $\partial G_0 / \partial n_\beta$ terms as shown next.

1.2 Derivation of the second kind Fredholm BIE for the Helmholtz equation

Since **r** is an interior point, we may generally differentiate beneath the integral sign. That is applying the operator $(\vec{n}_\beta \cdot \nabla_{\mathbf{r}})$ on equation (5) to obtain

$$\lim_{\mathbf{r}\rightarrow\beta} \frac{\partial}{\partial n_\beta} \psi(\mathbf{r}) =$$

$$\lim_{\mathbf{r}\rightarrow\beta} \int_{\partial D} \left(\frac{\partial}{\partial n_\beta} G_0(q, \mathbf{r}; k) \frac{\partial}{\partial n_q} \psi(q) - \psi(q) \frac{\partial}{\partial n_\beta} \frac{\partial}{\partial n_q} G_0(q, \mathbf{r}; k) \right) dq. \qquad (7)$$

Now the interior field point is positioned into the boundary $\mathbf{r} \rightarrow \beta \in \partial D$ in equation (7). This should not present any restrictions or difficulties and all the integrals remain well behaved as long as the source point is located far away from β. Imposing DBCs in equation (7) leads to,

$$\lim_{\mathbf{r}\rightarrow\beta} \frac{\partial}{\partial n_\beta} \psi(\mathbf{r}) = \lim_{\mathbf{r}\rightarrow\beta} \int_{\partial D} \frac{\partial}{\partial n_\beta} G_0(q, \mathbf{r}; k) \frac{\partial}{\partial n_q} \psi(q)\, dq. \qquad (8)$$

The RHS of equation (8) known in potential theory as the double layer potential, and there is a special relation for its limit to the boundary [4]. That is the kernel $\lim_{\boldsymbol{r}\rightarrow\beta} \frac{\partial}{\partial n_\beta} G_0(q, \boldsymbol{r}; k)$ has a jump when **r** tends to the boundary. Therefore, when formulating the BIEs, it is necessary to consider the discontinuity properties for the layer potentials [4, 7]. Take the limit $\mathbf{r} \rightarrow \beta \in \partial D$ and apply the jump condition [4] for equation (8), one obtains,

$$\mu(\beta) = 2 \int_{\partial D} \mu(q) \frac{\partial}{\partial n_\beta} G_0(q, \beta; k) dq\,, \qquad \mu(q) = \frac{\partial}{\partial n_q} \psi(q), \qquad (9)$$

,

This is a Fredholm equation of the second kind and its kernel is given as,

$$\frac{\partial}{\partial n_\beta} G_0(q, \beta; k) = \frac{ik}{2} \cos\theta(q, \beta)\, H_1^{(1)}(k|q - \beta|)$$

, (10)

and

$$\cos\theta(q,\beta) = \frac{(q-\beta).\vec{n}}{|q-\beta|}, \qquad \text{for} \quad |q-\beta| \neq 0 \tag{11}$$

where $\theta(q,\beta)$ is the angle between the normal at the boundary point β and the chord connecting the initial boundary point q to the final boundary point. For a very small argument of the Hankel function $H_1^{(1)}$ in the limit $q \to \beta$, one can use the following asymptotic expansion of the Hankel function as,

$$H_1^{(1)}(k|q-\beta|) \sim \frac{-2i}{\pi k|q-\beta|}, \qquad as\ |q-\beta| \to 0.$$

This asymptotic expansion of $H_1^{(1)}$ has a singularity of the order $\mathcal{O}\left(\frac{1}{r}\right)$ Nevertheless, such singularity is cancelled out by the geometric factor $\cos\theta(q,\beta)$. For a curved boundary, the term $\cos\theta(q,\beta)$ is obtained as,

$$\cos\theta(q,\beta) \sim \frac{1}{2}\kappa|q-\beta|, \qquad as\ |q-\beta| \to 0,$$

where κ is the boundary curvature, for a circle boundary, the curvature is the reciprocal of its radius. Thus for curved boundary, the kernel in equation (9) becomes

$$\begin{aligned} K(q,\beta;k) &= 2\frac{\partial}{\partial n_\beta} G_0(q,\beta;k), \\ &= \frac{ik}{2}\cos\theta(q,\beta)\, H_1^{(1)}(k|q-\beta|), \\ &\sim \frac{\kappa}{2\pi}, \quad as|q-\beta| \to 0\,. \end{aligned}$$

One needs to make use of the following property of the delta function,

$$\varphi(y) = \int_{-\infty}^{\infty} \delta(x-y)\varphi(x)dx\,. \tag{12}$$

So, the BIE (10) can be rewritten as,

$$\int_{\partial D}\left[\delta(q-\beta) - 2\frac{\partial}{\partial n_\beta} G_0(q,\beta;k)\right]\mu(q)dq = 0, \tag{13}$$

where the boundary elements q and β will be suppressed thereafter. This is a homogenous Fredholm integral equation of the second kind. After discretizing the boundary, equation (13) can be evaluated numerically to obtain a system of algebraic equations,

$$(I-K)\mu = 0.$$

Also we obtain the quantization condition as,

$$\det[I - K(q,\beta;k)] = 0\,. \tag{14}$$

The values of k which satisfy equation (14) are the eigenvalues. The notation I denotes the identity matrix and $K(q, \beta; k)$ is the boundary integral kernel defined for the boundary elements q and β. Since the matrix $(I - K)$ is antisymmetric and fully populated with non-zero coefficients, direct solvers such as Gaussian elimination should be used.

2. Discussion on the advantages of the normal derivative method.

Here we discuss the preference of using the second kind Fredholm equation over the first kind Fredholm equation for a unit disc. In principle, the Fredholm equation of first and second kinsd both are well adopted for numerical calculations [2]. An ill-conditioning problem is usually associated with Fredholm equations of the first kind, however this does not arise here. Because the presence of singularity of the kerenel ensures diagonal dominance in the system matrix, so the problem will be well-conditioned [2, 9]. The first advantage is that the BIE (6) leads to a symmetric matrix which is easy to handle, due to the symmetry property of the free-space Green function, whereas the BIE (9) leads to a non- symmetric matrix due to the term $\cos\theta$.

In the absence of absorbtion (real vlaues of the wavenumber k), the Green function must be real-valued as understood from equation (1) that the Laplace operator $\nabla^2$and the delta function $\delta(\mathbf{q} - \mathbf{r})$ are both real quantities. Therfore, only the real part of the Green function is contributed to the solution. The imaginary part of the boundary function μ must vanish or be very small. This fact will be used in varyfying our numerical computations of the boundary functions. Here we compare the accuracy of the boundary function μ computed from both BIEs for a unit disc with DBCs. Respectively, figure 1 and figure 2 show plots of the real and imaginary parts of the boundary functions μ , computed using both the original BIE (6) and the normal derivative BIE (9). Figure 1 shows that the real part of the boundary function $Re(\mu)$ computed using the original BIE (6) coincides very well with $Re(\mu)$ computed using the normal derivative BIE (9). As the imaginary part of μ gives the measure of the error, figure 2 shows that the normal derivative BIE (9) gives more accurate solution than the original BIE (6). Thus, we observe that the normal derivative BIE (9) is more efficient than the original BIE (6).

Furthermore, the quantization condition obtained from the first kind BIE (6) is not numerically stable and becomes exponenetially small as one increases the number of boundary elements. For instance, for 200 boundary elements the values of the determinant (14) are smaller than the machine underflow threshhold. Whereas the second kind Fredholm BIE (9) works efficiently for computing the spectrum. Table 1 shows the first ten of the sequence of the eigenvalues of a unit disc with DBCs which are the zeros of the quantization condition (14). These vlaues coincide very well with the analytic eigenvalues (zeros of Bessel functions) as tabulated in [6].

Table 1: The eigenvalues of the unit disc with DBCs within the k range [50.1, 51]

i	Eigenvalues
1	50.2453
2	50.5681
3	50.5836
4	50.6610
5	50.6782
6	50.8071
7	50.8438
8	50.9306
9	50.9377
10	50.9650

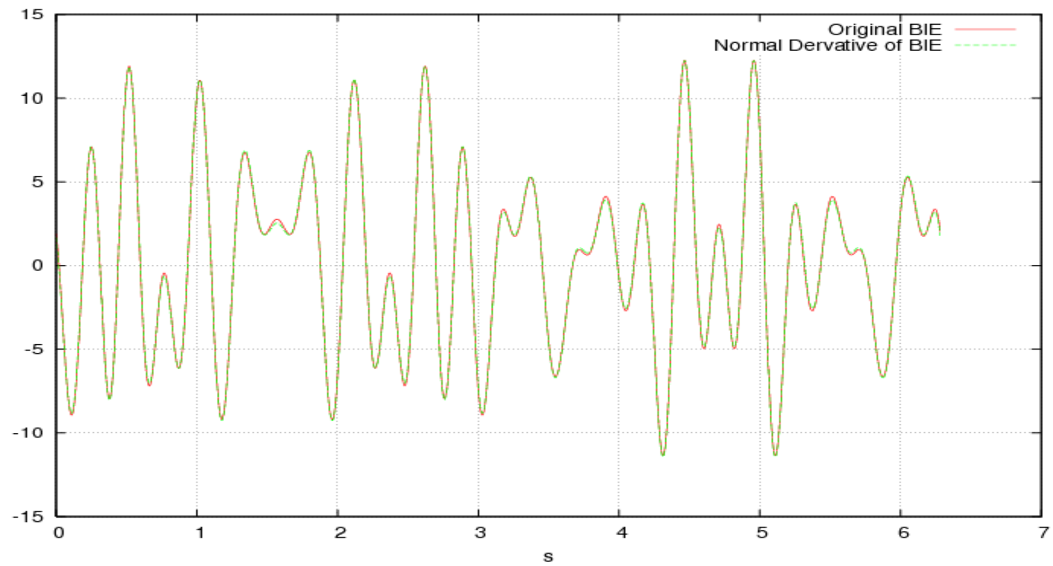

Figure 1 **Re(μ) for a unit disc with DBCs for k=50 and 2000 boundary elements**

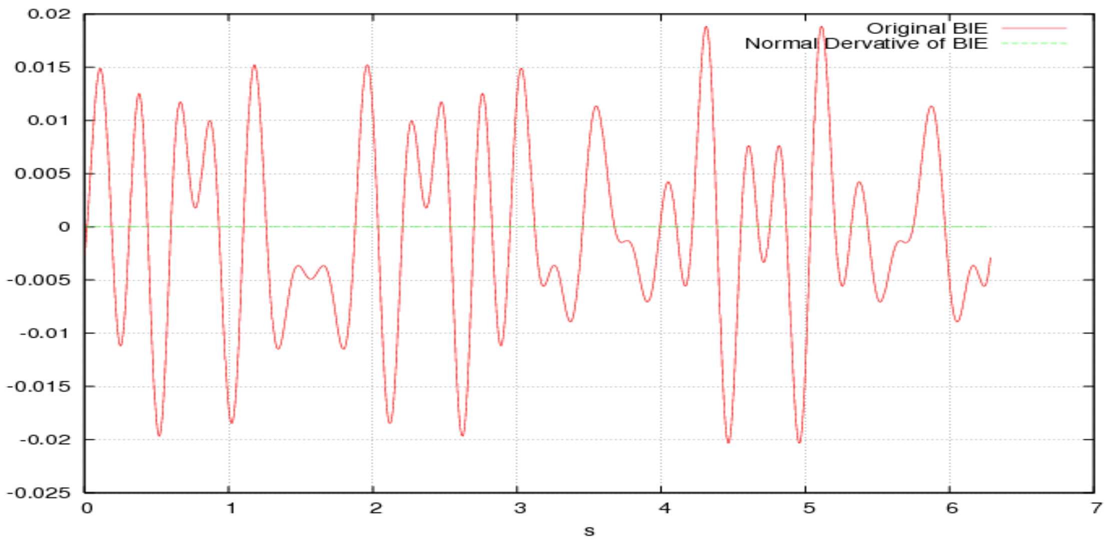

Figure 2 **Im(μ) for a unit disc with DBCs for k=50 and 2000 boundary elements**

3 The BEM formulation for the Green function

In this section we present the derivation of the Green function G which satisfies both the Helmholtz equation and the prescribed boundary conditions. Note that the free-space Green function G_0 is a solution of the Helmholtz equation but does not fulfil the prescribed boundary conditions. Since both of the Green functions G and G_0 satisfy the Helmholtz equation (1), one has

$$(\nabla^2 + \mathrm{k}^2)\mathrm{G}(\mathbf{q}, \acute{\boldsymbol{r}}; \mathrm{k}) = -\delta(\mathbf{q} - \acute{\boldsymbol{r}}) \,, \tag{15}$$

and

$$(\nabla^2 + \mathrm{k}^2)\mathrm{G}_0(\mathbf{q}, \mathbf{r}; \mathrm{k}) = -\delta(\mathbf{q} - \mathbf{r}) \,. \tag{16}$$

The wave-number k is defined as

$$k = \frac{\omega}{c} + i\eta, \qquad \eta > 0, \qquad i = \sqrt{-1}\,, \tag{17}$$

where ω is the angular frequency, the constants c and η represent respectively the wave velocity and the damping parameter of the system. Conceptually, the derivation of the Green function is similar to the derivation of the quantization condition (14). We begin by multiplying equation (15) and (16) by $\mathrm{G}_0(\mathbf{q}, \mathbf{r}; \mathrm{k})$ and $\mathrm{G}(\mathbf{q}, \acute{\boldsymbol{r}}; \mathrm{k})$, respectively, and taking the difference. Then we integrate the resulting equation over the domain D. Finally applying the Green second identity [1] and the prescribed DBCs to end up with this equation,

$$\mathrm{G}(\mathbf{r}, \acute{\boldsymbol{r}}) = \mathrm{G}_0(\mathbf{r}, \acute{\boldsymbol{r}}) + \int_{\partial D} \mathrm{G}_0(\mathrm{q}, \mathbf{r}; \mathrm{k})\, \mu(\mathrm{q}, \acute{\boldsymbol{r}}) \mathrm{dq}, \tag{18}$$

where

$$\mu(\mathrm{q},\acute{\boldsymbol{r}}) = \frac{\partial}{\partial \mathrm{n_q}} \mathrm{G}(\mathrm{q},\acute{\boldsymbol{r}};\mathrm{k}).$$

The Green function at any interior point **r** for a given source point $\acute{\boldsymbol{r}}$ can be computed using equation (18). The term $\mathrm{G_0}(\mathbf{r},\acute{\boldsymbol{r}})$ in equation (18) represents the direct contribution of rays starting at the source $\acute{\boldsymbol{r}}$ and reaching a receiver point **r** without any reflection from the boundary. The second term in equation (18) is called the single layer potential or the elementary potential with density $\mu(\mathrm{q},\acute{\boldsymbol{r}})$ [4]. Such a term represents the indirect contribution of rays which reach a receiver point after hitting the boundary at least once. Conceptually, it represents the correction to the free-space Green function to construct the full Green function with all reflections and propagation. The BEM analysis can be performed to find all the boundary unknowns and then in a post-processing procedures, by positioning the source point $\acute{\boldsymbol{r}}$ at the point of interest. The Green function can be calculated at any interior point by a numerical quadrature for the non-singular integrals in equation (18) using the boundary function $\mu(\mathrm{q},\acute{\boldsymbol{r}})$.

4. Conclusions

To summarize for a smooth boundary such as a disc, the normal derivative equation (9) works more efficiently than the original BIE (6). Built on this observation, we argue that the second kind integral equation (9) is more stable than the first kind integral equation (6). Also, we argue that the singularity of the free Green function $\mathrm{G_0}$ in the BIE (6) is responsible for the inaccuracy in computing the boundary functions and unstability issue encountered in computing the spectrum. However, for non-smooth geometries (edges and corners), the normal is not defined (not unique) at the corners. Hence corner corrections need to be considered as it may affect the accuracy [2]. Therefore the normal derivative method for non-smooth geometries is spoiled by the corners problem.

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