Some remarks on Rational Barycentric Interpolation

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Abstract

Linear barycentric rational interpolant are a specific type of rational interpolants, defined by weight independent of function. These interpolants have recently been a valuable alternative to more classical methods of interpolation. Rational interpolation gives a much better approximation than polynomial, but it is difficult to avoid poles and unattainable points.

In this paper , discuss the use of rational interpolation development, then we try to introduce another barycentric rational interpolant that provides a good result.

Keywords: *Polynomial interpolation, rational interpolation, barycentric interpolation.*

1. Introduction

The Polynomial interpolation is the dominant for approximation and has some clear advantages. For instance, any continuous function on a given interval [a, b] can be approximated by polynomials (*Weierstrass*). But there are some disadvantages, as a high polynomial degree is generally needed for accuracy, which in some cases leads to divergence. The rational interpolation is a promising alternative which can lead to better results in some cases. The presence of undesired poles and unattainable points near or inside of the interpolation interval can render it useless in such cases [7]. Barycentric rational interpolation as presented by Berrut and Mittelmann possesses many advantages over the classical rational interpolation. They showed that every rational interpolant may be written in the barycentric form. Floater and Horman also constructed the rational interpolant by blending polynomial interpolants and defined an explicit formula for weights [10]. The barycentric form of rational interpolants has many advantages over the polynomial one. In particular, it allows for an easier detection of unattainable points and of poles in the interval of interpolation.

Let $\{x_j\}_{j=0}^{n}$ be $n+1$ be an interpolation points in [a, b], where $a \le x_0 < x_1 < \ldots < x_n \le b$. The interpolation of the Lagrange polynomial can be defined by the formula[7]

$$p_n(x) = \sum_{j=0}^{n} f(x_j) L_J(x), \qquad L_J(x) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k} \qquad (1a)$$

Where

$$L_J(x_k) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

The formula of Lagrange interpolation is useful for theoretical purposes, but in practice it is not appropriate [7].

Let

$$L(x) = \prod_{k=0}^{n} (x - x_k) \ \rightarrow \ L'(x_j) = \prod_{\substack{k=0 \\ k \neq j}}^{n} (x_j - x_k).$$

Hence

$$L_J(x) = \frac{L(x)}{L'(x_j)(x - x_j)}.$$

If the weight is defined by

$$w_j = \frac{1}{\prod_{k=0}^{n}(x_j - x_k)} = \frac{1}{L'(x_j)}$$

then $L_J(x)$ can be written as

$$L_J(x) = L(x) \frac{w_j}{(x - x_j)}.$$

The modified formula for Lagrange is therefore defined as:

$$p_n(x) = L(x) \sum_{j=0}^{n} f_j \ \frac{w_j}{(x - x_j)} \qquad (1b)$$

If we interpolate $f_j = 1$ for all j is the unique polynomial $p_n(x) = 1$, which has zero degree. Hence, this implies that

$$1 = \sum_{j=0}^{n} L_J(x) = L(x) \sum_{j=0}^{n} \frac{w_j}{(x - x_j)}$$

So

$$L(x) = \frac{1}{\sum_{j=0}^{n} \frac{w_j}{(x - x_j)}}.$$

Therefore

$$p(x) = \frac{\sum_{j=0}^{n} f_j \frac{w_j}{x - x_j}}{\sum_{j=0}^{n} \frac{w_j}{x - x_j}}, \qquad (2)$$

This is known as the barycentric formula. It is a polynomial if the weights w_j are nonzero and defined in a such a way that

$$\sum_{j=0}^{n} L_j = L(x) \sum_{j=0}^{n} \frac{w_j}{(x - x_j)} = 1.$$

In 2004, Berrut and Trefethen discussed the formula. Since then for interpolation, the formula has been commonly known and used [7].

2. Classical Rational Interpolation

Let $\{x_j\}_{j=0}^{N}$ be distinct points and *f(x)* a given function. Let the polynomials *p(x)* of degree $\leq m$ and *q(x)* of degree $\leq n$, be the numerator and the denominator, respectively, where

$$r_{mn} = \frac{p(x)}{q(x)} = \frac{\sum_{k=0}^{m} a_k x^k}{\sum_{k=0}^{n} b_k x^k} \dots\dots\dots\dots\dots\dots\dots\dots\dots (3)$$

The rational interpolation interpolation problem is to find *r(x)* which satisfies the following condition [7]

$$r(x_j) = \frac{p(x_j)}{q(x_j)} = f(x_j), \qquad j = 0, \dots, m + n \dots\dots (4)$$

If r exists, the solution is unique and can be written as a barycentric formula. The condition for (4) to be satisfied is

$$q(x_j)f(x_j) - p(x_j) = 0 \dots\dots\dots\dots\dots\dots\dots (5)$$

Which is

$$a_0x_j + \dots + a_m x_j^m - f_j\left(a_0x_j + \cdots + a_m x_j^m\right) = 0$$

This is equivalent to a system of homogeneous equations, with $(m+1)$ unknown coefficients in $p(x)$ and $(n+1)$ in $q(x)$[8].

The system has at least one nonzero solution, where $N = n + m$, and $n \leq m$ without loss of generality, we have $m \geq \frac{N}{2}$.

It is well known that a polynomial interpolation of degree $\leq n$ exists and is unique. In classical rational interpolation, the condition $n + m = N$ is analogous to the condition $fq - p = 0$ for finit f and $f = \infty$ for $q = 0$. However, the rational interpolation does not always have a solution because of two major obstacles [3, 5]:

- In some cases the interpolation condition $r(x_j) = f(x_j)$ may not be achieved because of the occurrence of unattainable points. That is, there is a point x_j where the given function value f_j may not be obtained. This points x_j is called unattainable and occurs if x_j is a zero of the denominator $q_n(x)$ and the numerator $p_m(x)$.
- Rational interpolant may have poles in the interval of interpolation, which are zeros of $q_n(x)$that are not common to $p_m(x)$. Theses poles cause a problem if they are inside the interval of interpolation. Rational interpolation is useless if the function to be approximated is not singulat at the same points.

Example 1.1: Let x =1 and $x = 3$ are unattainable points: The corresponding solution $m = n = 2$ is

$$R(x) = \frac{2x^2 - 8x+6}{x^2 - 4x+3} = \frac{2(x-1)(x-3)}{(x-1)(x-3)} = 2.$$

The points $x = 1$ and x =3 are unattainable points: $R_{2,2}$ is $\frac{0}{0}$ at $x = 1$ and $x = 3$ and has the common factor $(x-1)(x-3)$, but $f = 2$. After cancellation $(x - x_j)$, $r(x_j)$ may be not equal to $f(x_j)$.

Example 1.2: Let $x_0 = -1$, $x_1 = 1$, $x_2 = 2$ and $f_0 = 2$, $f_1 = 3$, $f_2 = 3$. if we take n=m=1, then $R(x) = \frac{3x+3}{x+1} = 3$. The denominator is equal to zero at $x_0 = -1$ and so$R(x_0) = 3 \neq 2 = f_0$.

Therefore, the interpolation problem has no solution for the prescribed degree. But for n=0 and m=2, then $R(x) = \frac{36}{14-3x+x^2}$.

Remark: Consider unattainable point x_j for a non—trivial solution $\frac{p}{q} \in R_{n,m}$, where $p = q = 0$ after the cancellation of $x - x_j$ in $\frac{p}{q}$, we have $\frac{p_j}{q_j} \neq f_j$ [16].

3. Barycentric Rational Interpolation`

If the weights w_j are nonzero and defined in such way that $L(x) = \sum_{j=0}^{n} \frac{w_j}{x - x_j}$ is not equal to 1, then the formula (2) is rational. To show that the rational interpolates the function, the following lemma is proved:

Lemma: Let f_j be the value of the function at x_j, where *j=0…,n* with $x_j \neq x_k$ for $k \neq j$. Then if $u_{k \neq 0}$, the rational function

$$r(x) = \frac{\sum_{j=0}^{n} f_j \frac{u_j}{x - x_j}}{\sum_{j=0}^{n} \frac{u_j}{x - x_j}}, \qquad (6)$$

Interpolates f_k at x_k and $\lim_{x \to x_l} r(x) = f_l$.

Conversely, any rational interpolant function $r \in r_{nn}$ of f_j can be written in barycentric form using some weights [3].

Proof: Multiplying (6) by $\frac{L(x)}{L(x)}$

$$r(x) = \frac{\sum_{j=0}^{n} f_j \frac{u_j}{x - x_j}}{\sum_{j=0}^{n} \frac{u_j}{x - x_j}} \frac{\prod_{k=0}^{n}(x - x_k)}{\prod_{k=0}^{n}(x - x_k)}$$

$$= \frac{\sum_{j=0}^{n} w_j f_j \prod_{k=0}^{n}(x-x_k)}{\sum_{j=0}^{n} w_j \prod_{k=0}^{n}(x-x_k)}.$$

By taking the limit $\lim_{x \to x_l} r(x)$, we have

$$r(x_l) = \frac{\sum_{j=0}^{n} w_j f_j \prod_{k=0}^{n}(x_l - x_k)}{\sum_{j=0}^{n} w_j \prod_{k=0}^{n}(x_l - x_k)} = f_j.$$

The interpolation condition are satisfied as long as the interpolation points are distinct and w_j are not equal to zero.

From the Lagrange formula

$$p(x) = \sum_{j=0}^{n} f_j L_j(x),$$

The numerator p and the denominator q could be written as

$$r(x) = \frac{p(x)}{q(x)} = \frac{L(x) \sum_{j=0}^{n} \frac{p(x_j)}{(x - x_j)L'(x_j)}}{L(x) \sum_{j=0}^{n} \frac{q(x_j)}{(x - x_j)L'(x_j)}}$$

Let $r(x_j) = \frac{p(x_j)}{q(x_j)} = f_j$, where $r(x_j)\, q(x_j) = f(x_j)$. Then we have

$$r(x) = \frac{\sum_{j=0}^{n} f_j \frac{q(x_j)}{(x - x_j)L'(x_j)}}{\sum_{j=0}^{n} \frac{q(x_j) w_j}{(x - x_j)L'(x_j)}}$$

By defining the weights $u_j = \frac{q(x_j)}{L'(x_j)} = w_j\, q_j$, we arrive at

$$r(x) = \frac{\sum_{j=0}^{n} f_j \frac{u_j}{x - x_j}}{\sum_{j=0}^{n} \frac{u_j}{x - x_j}},$$

If *q(x) = 1*, then $q_j = 1$ for *j=0,…,n* and so (6) reduces to the barycentric form of the interpolating polynomial of degree n.

Berrut in [5] showed that if we use $w_j = (-1)^j$, then the interpolant has no poles in R, where

$$r(x) = \frac{\sum_{j=0}^{n} f_j \frac{(-1)^j}{x - x_j}}{\sum_{j=0}^{n} \frac{(-1)^j}{x - x_j}}, \dots \dots \dots \dots (8)$$

The advantages of rational interpolation over polynomials interpolation at equally spaced points investigated by many studies such in [3, 5, 7]. If we interpolate Runge's function, then the oscillation vanishes. Rational interpolation using Chebyshev points may yield worse results than using equally spaced points.

It is clear that barycentric rational interpolation allows one to choose the points and the weights.

The rational interpolation can be written in barycentric form, where $u_j = w_j q_j$ is the weight corresponding to the points x_j. Hence, the barycentric form has the advantage that the barycentric weights give information about possible unattainable points. The problem is how to choose the w_j to avoid poles and produce a good approximation.

It is hard to approximate functions with poles, but in general points they do not lead to a good approximation. For example, the weights of $w_j = \frac{1}{\prod_{k=0}^{n}(x_j - x_k)}$ avoid poles. However, the determination of the weight u_j is more complicated, which can be distinguished in specific ways:

Choosing the degree**:** This can be done by fixing the exact degree of the numerator and denominator of (6). By choosing the numerator of degree $\leq m$ and the denominator of degree $\leq n$, with $n + m + 1 = N$, such as

$$r(x) = \frac{\sum_{j=0}^{n} f_j \frac{u_j}{x - x_j}}{\sum_{j=0}^{n} \frac{u_j}{x - x_j}} = \frac{\sum_{0}^{m} a_j x^j}{\sum_{0}^{n} b_j x^j}$$

The a_j, b_j and u_j are unknown. The method has been studied in [3, 14, 16, 22].

Choosing the poles**:** This can be found by solving

$$\sum_{j=0}^{n} \frac{u_j}{x - x_j}.$$

If some of the poles are known, it is possible to attach them to the barycentric rational interpolant. These methods have been treated in [4].

In [3, 14, 16, 22], they have used a different basis to present q_n and p_{N-n} and derive a homogeneous system of linear equations in barycentric form. Schneider and Werner in [16] presented the rational interpolant in barycentric form with $n+1$ weights. They expressed q_n and p_{N-n} by using Lagrange basis:

$$p(x) = \sum_{j=0}^{n} u_j f_j \prod_{k=0}^{n} (x - x_k),$$

$$q(x) = \sum_{j=0}^{n} u_j \prod_{k=0}^{n} (x - x_k),$$

They suggested computing the weights by expressing the denominator $q(x)$ in a Newton basis

$$q(x) = \sum_{j=0}^{n} w_j \prod_{k=0}^{j-1} (x - x_k),$$

Then, by an algorithm of Werner in [19] unknown weights uj are found by computing *q(x)* at $x = x_j$, which is

$$u_j = \frac{q(x_j)}{\prod_{k=0}^{n} (x - x_k),}$$

The solution presented in [16] depends on the coefficients w_j, $0 \le j \le n$, $n \le m - N - n$ and solving the homogeneous linear system.

$$f[x_0, \dots, x_{m+1}]w_0 + \cdots + f[x_n, \dots x_{m+1}]w_n = 0$$

$$f[x_0, \dots, x_{m+2}]w_0 + \cdots + f[x_n, \dots x_{m+2}]w_n = 0$$

$$\vdots$$

$$f[x_0, \dots\dots, x_N]w_0 + \dots + f[x_n, \dots\dots x_N]w_n = 0$$

Then the barycentric form q_n, can be expressed as

$$q_n(x) = \prod_{k=0}^{n}(x - x_k) \sum_{j=0}^{n} \frac{w_j}{x - x_j} q_{j,}$$

$$w_j = \frac{1}{\prod_{k=0}^{n}(x_l - x_k)}.$$

For the resulting linear system, the matrix of coefficients is a divided difference.

A direct method due to Berrut and Mittelmann for finding the corresponding weights u_j can be found in [3]. Using monomial basis functions, conditions for the weights are derived such that $r_n(x) = \frac{p}{q}$, where the degree of p and q are m and n, respectively. A very similar approach is due to Zhu in [22] and Polezzi in [15] who directly determined function values q_n of a denominator of degree at most n from the same degree condition. Schneider and Werner in [16] proposition stated that:

Proposition: [4, 16] If the rational interpolant r has no poles, then

$$w_j \,.\, w_{j+1} < 0, \;\; j = 0, \dots, N.$$

This condition is necessary, but is not sufficient. A sufficient is still an open problem, despite some effort has been made.

Example 1.3: Let *x*=[-1, 0, 1] and *u*=[1, -1, 6] where *n*=2. The alternating the sign u_j, *j=0, 1, 2* implies the absence of poles.

Example 1.4: For *x*=[1, 1,5, 3] and *w*=[1, 1, 1] where n=2, f_j*=1, 0.7, 1.7*, the function $r_2(x)$ has poles.

Example 1.5: For *x*=[1, 1,5, 3] and *w*=[1, -1, 1] where *n=2,* f_j*=1, 0.7, 1.7*, the function $r_2(x)$ does not have the poles.

A sufficient condition is that the u_j are similar enough in in sizes for the $\frac{u_j}{x-x_j}$ to decrease in absolute value on both size for each x [5, Lemma2.1].

Proposition: [3, 16] If for some j,

$$w_j \,.\, w_{j+1} > 0, \;\; j = 0, \ldots, N.$$

Then r has an odd number of poles in $[x_j, x_{j+1}]$.

Remark: We can note that u_j in a rational presentation oscillates in sign, because the equation

$$u_j = w_j q_{j,}$$

Implies that w_j oscillate in sign but q_j does not.

An implicit form of (6) can be written as

$$\sum_{j=0}^{n} u_j \frac{r_n(x) - f_j}{x - x_j} = 0$$

4. Floater and Horman interpolant

An attractive rational interpolation method that has no poles is presented in [10] by Floater and Hormann. They constructed an interpolant by blending the polynomial interpolants. Consider $n + 1$ points $x_0 < x_1 \ldots.. < x_n$ with corresponding values $f_0 < f_1 \ldots.. < f_n$. Suppose that $0 \leq d \leq n$. For $j = 0, \ldots, n - d$, let p_j be the polynomial of degree at most d that interpolates $x_j, \ldots\ldots, x_{j+d}$. Then interpolant undertaken in [10] is

$$R_n(x) = \frac{\sum_{j=0}^{n-d} \lambda_j(x) p_j(x)}{\sum_{j=0}^{n-d} \lambda_j(x)} \ldots\ldots\ldots.(9)$$

$$\lambda_j(x) = \frac{(-1)^j}{(x - x_j) \dots (x - x_{j+d})}$$

The interpolant (9) has no real poles for each d, and has an approximation of order $O(h^{d+1})$ *as* $h \to 0$, where h is a maximum of the space between two adjacent points

$$h = \max_{0 \le j \le n-1} (x_{j+1} - x_j),$$

for $f(x) \in C^{d+2}$*[a,b]*.

The formula (9) is expensive to evaluate, but if we multiply the numerator and denominator by $L(x)$ we see that (9) is a rational function of degree at most n and $n - d$. So (9) can be written in the barycentric form (6). The weights for the barycentric form $r_n(x)$ of (9) are defined explicitly:

$$u_j = \sum (-1)^j \frac{1}{x_j - x_k}.$$

This because the (9) has no poles and so its weights oscillate in sign [10]. The d gives rise to a whole family of interpolants with no poles and high approximation orders. In theory, the approximation error decrease as d increases. In practice, however due to finite precision arithmetic, the numerically computed approximation may not behave accordingly. For moderate d, the interpolant (9) performs very well even for large n.

5. An Improved Floater and Hofmann Interpolant

The Floater and Hormann interpolation formula faces a problem in the cases where $d = n$ or $d = 0$. In this case, the interpolant r will be interpolating polynomials . The idea of the proposed formula is that instead of blending the polynomial interpolants, the rational interpolants will be blended. Therefore when $d = n$ or 0 the interpolant will be rational interpolations.

For $0 \le d \le n$, let r_j be the barycentric rational interpolation that interpolates $x_j, \dots, x_{j+d}$. The interpolant undertaken based on Floater and Hormann is

$$R_n(x) = \frac{\sum_{j=0}^{n-d} \lambda_j(x) r_j(x)}{\sum_{j=0}^{n-d} \lambda_j(x)} \dots\dots\dots\dots\dots (10)$$

$$\lambda_j(x) = \frac{(-1)^j}{(x - x_j) \dots (x - x_{j+d})} \dots\dots.\dots (11)$$

The formula (10) has no poles and so its weights oscillate in sign.

Theorem: Let $\{x_j\}_{j=0}^{N}$ be $N + 1$ distinct points. Then the rational interpolant defined by (10) satisfies the interpolation condition $r(x_k) = f(x_k), k = 0, 1, \dots, N$.

Proof: By multiplying the numerator and denominator by $(-1)^{n-d} (x - x_0) \dots (x - x_n)$, we have

$$r_n(x) = \frac{\sum_{j=0}^{n-d} (-1)^{n-d} (x - x_0) \dots (x - x_n) \lambda_j(x) r_j(x)}{\sum_{j=0}^{n-d} (-1)^{n-d} (x - x_0) \dots (x - x_n) \lambda_j(x)}$$

Letting $\mu_j(x) = (-1)^{n-d} (x - x_0) \dots (x - x_n) \lambda_j(x)$, then

$$\lim_{x \to x_k} r(x) = \frac{\sum_{j=0}^{n-d} \mu_j(x) r_j(x)}{\sum_{j=0}^{n-d} \mu_j(x)}$$

We know from Lemma 1.5 the rational function satisfies $\lim_{x \to x_k} r(x_k) = f(x_k)$. Then

$$r(x_k) = \frac{\sum_{j=0}^{n-d} (-1)^{n-d} \mu_j(x_k) f(x_k)}{\sum_{j=0}^{n-d} (-1)^{n-d} \mu_j(x_k)} = f(x_k).$$

Theorem: Let the function $f \in C^{d+2}[a, b]$, the rational $r_n(x)$ is obtained by (10), then when $n - d$ is odd, we have

$$\|f(x) - R_n(x)\| \leq h^{d+1}\,(1+\beta)(b-a)\frac{\|f^{d+2}\|}{d+2},$$

And if $n - d$ is even

$$\|f(x) - R_n(x)\| \leq h^{d+1}(1+\beta)(b-a)\left(\frac{\|f^{d+2}\|}{d+2} + \frac{\|f^{d+1}\|}{d+1}\right),$$

$$h = \max_{0\leq i\leq n-1}(x_{i+1} - x_i).$$

$$\beta = \max_{1\leq l\leq n-1} \min\left\{\frac{x_{i+1} - x_i}{x_i - x_{i-1}}, \frac{x_{i+1} - x_i}{x_{i+2} - x_{i+1}}\right\}$$

Proof: We start as in [11] for x which is an interpolation points, the error will be zero. And for *x* is not an interpolation points, the error can be presented as:

$$\|f(x) - r_n(x)\| = \left|\frac{\sum_{j=0}^{n-d}\frac{(-1)^j}{(x-x_j)\ldots(x-x_{j+d})}\left(f(x) - r_j(x)\right)}{\sum_{j=0}^{n-d}\frac{(-1)^j}{(x-x_j)\ldots(x-x_{j+d})}}\right| \qquad \ldots\ldots\ldots\ldots (12)$$

Using

$$\left\|f - \frac{p_n}{q_n}\right\| \leq \max_{x\in[a,b]}\left(\frac{|(f\,q_m)^{n+1}(x)|}{(n+1)!}\right)\max_{x\in[a,b]}\frac{\prod_{k=0}^{n}|x - x_k|}{|q_m(x)|}.$$

In [10], we have

$$(-1)^{n-d}\,(x - x_0)\ldots.(x - x_n)\,\lambda_i(x),\mu_i(x) =$$

$$\lambda_i(x) = \frac{\mu_i(x)}{(-1)^{n-d}\,(x - x_0)\ldots.(x - x_n)}.$$

Then, we can see that, the numerator and the denominator of (12) is multiplied by $(-1)^{n-d}$ $(x - x_0)\ldots.(x - x_n)$, which is independent of i . From [10], we have

$$\sum_{i=0}^{n-d} \mu_i (x) > 0.$$

Now, from (12), we can see the error $f(x) - r_j(x)$ that related to the formula (8). Therefore, we have the above.

To prove the case when $n - d$ is even, we follow the similar procedure of an odd $n - d$.

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