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Oscillations Of Solutions For Nonlinear Differential Equations

Jaffalah J. Amhalhil

e-mail: jaffalah.ali@su.edu.ly

Sirte University, School of Mathematical Sciences, Faculty of Education.

Abstract

It is know that many problems in physics, in the study of chemically reacting systems, in celestial mechanics and in other fields of science can be modelled by second order nonlinear differential equations. Therefore, the asymptotic and oscillatory properties of solutions of such equations have been investigated by many authors. In this paper our aim is to present some new sufficient conditions for the oscillation of all solutions of the nonlinear differential equations of the form

$$\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\bullet}+g_1(t,x(t))=0 \qquad (1)$$

Our new results extend and improve a number of existing oscillation criteria. Further, Our main results are illustrated with examples.

Keywords: *Oscillation criteria, ordinary differential equations, Second order, Nonlinear*

1. Introduction

The study of the oscillation of second – order nonlinear ordinary differential equations with alternating coefficients is of special interest because many physical system are modelled by second – order nonlinear ordinary differential equations, for example, the so – called Emden - Fowler equation arises in the study of gas dynamics and fluid mechanics. This equation appears also in the study of relativistic mechanics and in the study of certain chemical reactions.

The problem of determining oscillation criteria for second – order nonlinear differential equations has received a great deal of attention in the twenty years after the publication of the classic paper by Atkinson [1].

Many authors use some different techniques in studying the oscillatory behaviour of the second order differential equations, especially, what so – called averaging techniques that dates back to works of Wintner [15] and its generalization by Hartman [8].

Investigation of the differential equation (1) in this paper is motivated by the paper [3], where some of the conditions required in the theorems contain the unknown solution . It seems that any verification of such conditions is questionable and may be impossible. The purpose of the paper is to remove the above mentioned conditions that depend on solution and improve some results presented in [3] in this way. The relevance of theorems in the text is illustrated by included examples. In the last time increases the number of papers which involve oscillatory criteria based on the idea of using of the parameter functions $H(t,s)$ (see e.g. [9-11]). These results have great theoretical value but they are less effective in applications. On the other hand, the results which contain the requirements only on the functions occurring in differential equation are usually better applicable. The paper contains only results of the latter kind.

In this paper we shall study the oscillatory behaviour of the solution of the differential equation of the form

$$\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\bullet} + g_1(t,x(t)) = 0 \qquad (1)$$

Where r is a positive continuous function on the interval $[t_0,\infty), t_0 \geq 0$, ψ is a positive continuous function on the real line R and g_1 is a continuous function on $R \times R$, with

$\dfrac{g_1(t,x(t))}{g(x(t))} \geq q(t),\ \ for\ all\ \ x \neq 0$ and $t \in [t_0,\infty)$, where g is continuously differentiable function on the real line R except possible at 0 with $xg(x) > 0$, $g'(x) \geq l > 0\ \ for\ all\ \ x \neq 0$ and q is a continuous functions on the interval $[t_0,\infty), t_0 \geq 0$.

Throughout this paper we restrict our attention only to the solution of the differential equation (1) which exists on some interval $[t_0,\infty), t_0 \geq 0$ may depend on a particular solution.

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In the present section we shall state and prove some sufficient oscillation criteria of the solutions of the equation (1).

2. Main Results:

Theorem 2.1: Suppose that

$O_1: \quad \frac{1}{\psi(x)} \geq l_1 > 0, \quad for\ all\ \ x \in R,$

$O_2: \quad \lim_{t\to\infty} \frac{1}{R(t)} = k_1 \in [0,\infty)$, where $R(t) = \int_{t_0}^{t} \frac{ds}{r(s)} \quad for\ \ t > t_0$.

Furthermore , let for some integer $n \geq 3$

$O_3: \lim_{t\to\infty} \sup \frac{1}{R^{n-1}(t)} \int_{t_0}^{t} [R(t) - R(s)]^{n-1} q(s) ds = \infty,$

Then equation (1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of the differential equation (1) and that $x(t) \neq 0, \ \ for\ t \geq T_2 \geq t_0$,

Define
$$\omega(t) = \frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}, \qquad t \geq T_2 , \tag{2}$$

Thus by (1) it follows that

$$\dot{\omega}(t) = \frac{-g_1(t, x(t))}{g(x(t))} - \frac{\omega^2(t) g'(x(t))}{r(t)\psi(x(t))} \tag{3}$$

From the condition (O_1), for all $t \geq T_2$, we obtain

$$\dot{\omega}(t) \le q(t) - k\frac{\omega^2(t)}{r(t)}, \qquad t \ge T_2, \text{ where } k = ll_1 \text{ is a positive constant.}$$

Hence, for every $t \ge T_2$, we have

$$\int_{T_2}^{t} [R(t)-R(s)]^{n-1} q(s)ds \le -\int_{T_2}^{t} [R(t)-R(s)]^{n-1} \dot{\omega}(s)ds - k\int_{T_2}^{t} [R(t)-R(s)]^{n-1} \frac{\omega^2(s)}{r(s)}ds$$

$$\le [R(t)-R(T_2)]^{n-1}\omega(T_2) - \int_{T_2}^{t} k[R(t)-R(s)]^{n-1}\frac{\omega^2(s)}{r(s)}ds$$

$$-\int_{T_2}^{t} (n-1)[R(t)-R(s)]^{n-2}\frac{\omega(s)}{r(s)}ds \qquad (4)$$

$$\le [R(t)-R(T_2)]^{n-1}\omega(T_2) + \frac{(n-1)^2}{4k}\int_{T_2}^{t}\frac{[R(t)-R(s)]^{n-3}}{r(s)}ds$$

$$-\int_{T_2}^{t}\left\{\sqrt{\frac{k[R(t)-R(s)]^{n-1}}{r(s)}}\,\omega(s) + \frac{(n-1)}{2\sqrt{k}}\sqrt{\frac{[R(t)-R(s)]^{n-3}}{r(s)}}\right\}^2 ds,$$

Then, for $t \ge T_2$, we have

$$\int_{T_2}^{t} [R(t)-R(s)]^{n-1} q(s)ds \le [R(t)-R(T_2)]^{n-1}\omega(T_2) + \frac{(n-1)^2}{4k(n-2)}[R(t)-R(T_2)]^{n-2}. \qquad (5)$$

Then we have

$$\frac{1}{R^{n-1}(t)}\int_{t_0}^{t}[R(t)-R(s)]^{n-1}q(s)ds = \frac{1}{R^{n-1}(t)}\int_{t_0}^{T_2}[R(t)-R(s)]^{n-1}q(s)ds$$

$$+\frac{1}{R^{n-1}(t)}\int_{T_2}^{t}[R(t)-R(s)]^{n-1}q(s)ds$$

$$\le \frac{1}{R^{n-1}(t)}\int_{t_0}^{T_2}R^{n-1}(t)q(s)ds + \left(1-\frac{R(T_2)}{R(t)}\right)^{n-1}\omega(T_2) + \frac{(n-1)^2}{4k(n-2)R(t)}\left(1-\frac{R(T_2)}{R(t)}\right)^{n-2}, \qquad (6)$$

.

It is clear that for $t \in [t_0, T_2]$ it holds

$$\frac{1}{R^{n-1}(t)}\int_{t_0}^{T_2}[R(t)-R(s)]^{n-1}q(s)ds \le \frac{1}{R^{n-1}(t)}\int_{t_0}^{T_2}R^{n-1}(t)q(s)ds = \int_{t_0}^{T_2}q(s)ds \tag{7}$$

Combining (6) and (7) it follows that

$$\frac{1}{R^{n-1}(t)}\int_{t_0}^{t}[R(t)-R(s)]^{n-1}q(s)ds \le \left[1-\frac{R(T_2)}{R(t)}\right]^{n-1}\omega(T_2)+\frac{(n-1)^2}{4k(n-2)R(t)}\left[1-\frac{R(T_2)}{R(t)}\right]^{n-2}+\int_{t_0}^{T_2}q(s)ds\,,$$

It is clear that $\lim_{t\to\infty}\frac{1}{R(t)} = k_1 \in [0,\infty)$, thus

$$\limsup_{t\to\infty}\frac{1}{R^{n-1}(t)}\int_{t_0}^{t}[R(t)-R(s)]^{n-1}q(s)ds \le [1-k_1R(T_2)]^{n-1}\omega(T_2)$$

$$+\frac{k_1(n-1)^2}{4k(n-2)}[1-k_1R(T_2)]^{n-2}+\int_{t_0}^{T_2}q(s)ds<\infty,$$

This contradicts to the condition (O_3); hence, the proof is completed.

Example 2.1: Consider the equation

$$\left[\left(\frac{1}{t}\right)\left(\frac{1+x^4(t)}{2+x^4(t)}\right)\dot{x}(t)\right]^{\bullet}+x^3(t)\left(1+x^2(t)\right)=0\,,\quad t>0 \tag{8}$$

Notice that

$$r(t)=\frac{1}{t}>0 \quad \forall t\ge t_0>0\,;\ \psi(x)=\frac{1+x^4}{2+x^4}>0 \quad and \quad \frac{1}{\psi(x)}=\frac{2+x^4}{1+x^4}\ge 1 \quad \forall\ x\in R\,,$$

$$\frac{g_1(t,x(t))}{g(x(t))}=\frac{x^3(t)\left(1+x^2(t)\right)}{x^3(t)}=1+x^2(t)\ge 1=q(t) \qquad for\ \ all\ \ x\ne 0\ \ and\ \ t\in[t_0,\infty)$$

$$R(t)=\int_{t_0}^{t}\frac{ds}{r(s)}=\int_{t_0}^{t}sds=\frac{s^2}{2}\Big|_{t_0}^{t}=\frac{t^2}{2}-\frac{t_0^2}{2}\quad , \ for\ \ t>0 \quad and \quad \lim_{t\to\infty}\frac{1}{R(t)}=0\ \ , t\in[0,\infty),$$

Let n=3 , then

$$\lim_{t\to\infty}\sup\frac{1}{R^{n-1}(t)}\int_{t_0}^{t}\left[R(t)-R(s)\right]^{n-1}q(s)ds=\lim_{t\to\infty}\sup\frac{4}{(t^2-t_0^2)^2}\int_{t_0}^{t}\left[\frac{t^2}{2}-\frac{s^2}{2}\right]^2 ds=\infty.$$

By theorem 2.1, every solution of equation (8) oscillates.

Remark 2.1: Theorem 2.1 extends the results of E. M.Elabbasy and Sh. R. Elzeiny [2] and Ohriska and A.Zulova [12].

Theorem 2.2: Suppose that (O_1) holds. And furthermore let for some integer $n\geq 2$

$$O_4:\ \lim_{t\to\infty}\sup\frac{1}{t^n}\int_{t_0}^{t}\left[(t-s)^n q(s)-\frac{n^2}{4k_2}(t-s)^{n-2}r(s)\right]ds=\infty,$$

Then equation (1) is oscillatory .

Proof. Let $x(t)$ be a non-oscillatory solution of equation (1) and that $x(t)\neq 0$ for $t\geq T_3\geq t_0$.

Define
$$\omega(t)=\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))},\quad t\geq T_3,$$

Then, for every $t\geq T_3$, we obtain

$$\dot{\omega}(t)\leq -q(t)-k\frac{\omega^2(t)}{r(t)},\qquad t\geq T_3$$

Hence, for every $t\geq T_3$, we get

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$$\int_{T_3}^{t}(t-s)^n q(s)ds \le (t-T_3)^n \omega(T_3) + \frac{n^2}{4k}\int_{T_3}^{t}(t-s)^{n-2} r(s)ds$$

$$-\int_{T_3}^{t}\left\{\sqrt{\frac{k(t-s)^n}{r(s)}}\,\omega(s) + \frac{n}{2}\sqrt{\frac{r(s)(t-s)^{n-2}}{k}}\right\}^2 ds,$$

$$\le (t-T_3)^n \omega(T_3) + \frac{n^2}{4k}\int_{T_3}^{t}(t-s)^{n-2} r(s)ds\ , \tag{9}$$

Then , for all $t \ge T_3$, we have

$$\int_{T_3}^{t}\left[(t-s)^n q(s) - \frac{n^2 r(s)}{4k}(t-s)^{n-2}\right]ds \le (t-T_3)^n \omega(T_3)$$
$$\le (t-t_0)^n \omega(T_3), \qquad t \ge T_3\ . \tag{10}$$

Using the inequality (10) we get for $t \ge t_0$

$$\int_{t_0}^{t}\left[(t-s)^n q(s) - \frac{n^2}{4k}(t-s)^{n-2} r(s)\right]ds = \int_{t_0}^{T_3}\left[(t-s)^n q(s) - \frac{n^2}{4k}(t-s)^{n-2} r(s)\right]ds$$

$$+\int_{T_3}^{t}\left[(t-s)^n q(s) - \frac{n^2}{4k}(t-s)^{n-2} r(s)\right]ds$$

$$\le \int_{t_0}^{T_3}(t-s)^n q(s)ds + (t-t_0)^n \omega(T_3)$$

By the Bonnet theorem for a fixed $c_s \in \lfloor t_0, T_3 \rfloor$ such that

$$\int_{t_0}^{T_3}(t-s)^n q(s)ds = (t-t_0)^n \int_{t_0}^{c_s} q(s)ds.$$

Then , for $t \ge T_3$, we get

$$\int_{t_0}^{t}\left[(t-s)^n q(s)-\frac{n^2 r(s)}{4k}(t-s)^{n-2}\right]ds \le (t-t_0)^n\int_{t_0}^{c_s} q(s)ds+(t-t_0)^n\omega(T_3)\ . \qquad (11)$$

Now if we divide (11) by t^n take the upper limit as $t\to\infty$, we get

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}\left[(t-s)^n q(s)-\frac{n^2}{4k}(t-s)^{n-2}r(s)\right]ds<\infty\ .$$

This contradicts to the condition (O_4); hence, the proof is completed.

Example 2.2: Consider the equation

$$\left[\left(\frac{x^2(t)+2}{x^2(t)+4}\right)\dot{x}(t)\right]^{\bullet}+x(t)(1+x^4(t))=0\ ,\ t>0 \qquad (12)$$

Observe that

$$r(t)=1>0\ ,\quad t\ge t_0>0\ ,\quad \psi(x)=\frac{x^2+2}{x^2+4}>0\quad ,\ and\quad \frac{1}{\psi(x)}=\frac{x^2+4}{x^2+2}\ge 1\quad ,\ \forall x\in R\ ,$$

$$\frac{g_1(t,x(t))}{g(x(t))}=\frac{x(t)\left(1+x^4(t)\right)}{x(t)}=1+x^4(t)\ge 1=q(t)\quad ,\ for\ \ x\ne 0\ \ and\ \ t\in[t_0,\infty)\ ,$$

And for any integer $n\ge 2$, we have

$$\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}\left[(t-s)^n-\frac{n^2}{4k_2}(t-s)^{n-2}r(s)\right]ds=\limsup_{t\to\infty}\frac{1}{t^n}\int_{t_0}^{t}\left[(t-s)^n-\frac{n^2}{4}(t-s)^{n-2}\right]ds=\infty.$$

It follows from theorem 2.2 that given every solution of equation (12) is oscillates.

Remark 2.2: Theorem 2.2 extends the results of Ohriska and A.Zulova [12].

Theorem 2.3 : Suppose that

$O_5:\ 0<l_2\le\psi(x(t))\le l_3\quad ,\ for\ \ all\ \ x\in R\ ,$

And moreover, assume that there exists a differentiable function

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$\rho:[t_0,\infty)\to(0,\infty)$, and the continuous functions $h\,,\,H:D\equiv\{(t,s):t\ge s\ge t_0\}\to R$,

Where H has a continuous and non-positive partial derivative on D with respect to the second variable such that $H(t,t)=0$, *for* $t\ge t_0$, $H(t,s)>0$, *for* $t>s\ge t_0$, and

$$\frac{-\partial H(t,s)}{\partial s}=h(t,s)\sqrt{H(t,s)}\ , \qquad \textit{for all}\ \ (t,s)\in D\ ,$$

$$O_6:\ \limsup_{t\to\infty}\left[X(t,t_0)-\frac{1}{4c_1}Y(t,t_0)\right]=\infty\ ,\quad \text{Where}\quad X(t,t_0)=\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)ds\ ,$$

And $Y(t,t_0)=\dfrac{1}{H(t,t_0)}\displaystyle\int_{t_0}^{t}r(s)\rho(s)\left[\gamma(s)\sqrt{H(t,s)}+l_3h(t,s)\right]^2 ds$,

Then equation (1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of equation (1) and assume that $x(t)>0$ *for all* $t\ge T_1\ge t_0$.

Define
$$\omega(t)=\rho(t)\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}\ ,\ \ t\ge T_1\,, \tag{13}$$

Then , for every $t\ge T_1$, we obtain

$$\dot{\omega}(t)=-\frac{\rho(t)g_1(t,x(t))}{g(x(t))}+\frac{\dot{\rho}(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}-\frac{\rho(t)r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))}\,.$$

Therefore, for all $t\ge T_1$, we have

$$\dot{\omega}(t)\le-\rho(t)q(t)-\frac{1}{l_3}\left[\frac{l}{r(t)\rho(t)}\omega^2(t)+\gamma(t)\omega(t)\right]\ ,\ \ t\ge T_1\qquad ,\qquad \text{where}\ \ \gamma(t)=-\frac{l_3\dot{\rho}(t)}{\rho(t)}\,.$$

Then, for all $t\ge T_1\ge t_0$, we obtain

$$\int_{T_1}^{t} H(t,s)\rho(s)q(s)ds \le -\int_{T_1}^{t} H(t,s)\dot{\omega}(s)ds - \frac{1}{l_3}\int_{T_1}^{t}\left[\frac{lH(t,s)}{r(s)\rho(s)}\omega^2(s) + \gamma(s)H(t,s)\omega(s)\right]ds\ .$$

$$\therefore \int_{T_1}^{t} H(t,s)\rho(s)q(s)ds \le \left[H(t,s)\omega(s)\Big|_{T_1}^{t} - \int_{T_1}^{t}\frac{\partial H(t,s)}{\partial s}\omega(s)ds\right]$$

$$-\frac{1}{l_3}\int_{T_1}^{t}\left[\frac{lH(t,s)}{r(s)\rho(s)}\omega^2(s) + \gamma(s)H(t,s)\omega(s)\right]ds$$

Then, for $t \ge T_1$, we have

$$\int_{T_1}^{t} H(t,s)\rho(s)q(s)ds \le H(t,T_1)\omega(T_1) - \int_{T_1}^{t} h(t,s)\sqrt{H(t,s)}\omega(s)ds$$

$$-\frac{1}{l_3}\int_{T_1}^{t}\left[\frac{lH(t,s)}{r(s)\rho(s)}\omega^2(s) + \gamma(s)H(t,s)\omega(s)\right]ds$$

$$\le H(t,T_1)\omega(T_1) - \frac{1}{l_3}\int_{T_1}^{t}\left\{\frac{lH(t,s)}{r(s)\rho(s)}\omega^2(s) + \left[l_3h(t,s)\sqrt{H(t,s)} + \gamma(s)H(t,s)\right]\omega(s)\right\}ds$$

$$\le H(t,T_1)\omega(T_1) + \int_{T_1}^{t}\frac{r(s)\rho(s)}{4ll_3}\left[\gamma(s)\sqrt{H(t,s)} + l_3h(t,s)\right]^2 ds$$

$$-\frac{1}{l_3}\int_{T_1}^{t}\left[\sqrt{\frac{lH(t,s)}{r(s)\rho(s)}}\,\omega(s) + \sqrt{\frac{r(s)\rho(s)}{4l}}\left\langle l_3h(t,s) + \gamma(s)\sqrt{H(t,s)}\right\rangle\right]^2 ds$$

$$\le H(t,T_1)\left[\omega(T_1) - J(t,T_1)\right] + \int_{T_1}^{t}\frac{r(s)\rho(s)}{4c_1}\left[\gamma(s)\sqrt{H(t,s)} + l_3h(t,s)\right]^2 ds, \tag{14}$$

Where $$J(t,T_1) = \frac{1}{l_3H(t,T_1)}\int_{T_1}^{t}\left[\sqrt{\frac{lH(t,s)}{r(s)\rho(s)}}\,\omega(s) + \sqrt{\frac{r(s)\rho(s)}{4l}}\left\langle l_3h(t,s) + \gamma(s)\sqrt{H(t,s)}\right\rangle\right]^2 ds$$

And $c_1 = ll_3$ is a positive constant,

Moreover (14) implies that for $t \ge T_1$, we have

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$$\int_{T_1}^{t} H(t,s)\rho(s)q(s)ds - \frac{1}{4c_1}\int_{T_1}^{t} r(s)\rho(s)\left[\gamma(s)\sqrt{H(t,s)} + l_3 h(t,s)\right]^2 ds \le H(t,T_1)\omega(T_1),$$

Then, for $t \ge T_1$, we get

$$\le H(t,t_0)\left|\omega(T_1)\right|, \qquad for\ T_1 \ge t_0\ . \tag{15}$$

In view of (14) and (15) we can easily obtain that

$$H(t,t_0)\left[X(t,t_0) - \frac{1}{4c_1}Y(t,t_0)\right] = \int_{t_0}^{T_1}\left\{H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4c_1}\left[\gamma(s)\sqrt{H(t,s)} + l_3 h(t,s)\right]^2\right\}ds$$
$$+ \int_{T_1}^{t}\left\{H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4c_1}\left[\gamma(s)\sqrt{H(t,s)} + l_3 h(t,s)\right]^2\right\}ds$$

$$\le H(t,t_0)\int_{t_0}^{T_1}\left|\rho(s)q(s)\right|ds + H(t,t_0)\left|\omega(T_1)\right|,$$

For $t \ge T_1$, and so we have

$$\limsup_{t\to\infty}\left[X(t,t_0) - \frac{1}{4c_1}Y(t,t_0)\right] \le \int_{t_0}^{T_1}\left|\rho(s)q(s)\right|ds + \left|\omega(T_1)\right|.$$

Since this last inequality contradicts the condition (O_6), the proof is completed.

Example 2.3: Consider the equation

$$\left[t\left(6 + \frac{x^2(t)}{1+x^2(t)}\right)\dot{x}(t)\right]^{\bullet} + x(t)\left(t^2 + x^4(t)\right) = 0\ , \quad t > 0 \tag{16}$$

We note that

$$r(t)=t>0\ ,\quad t\ge t_0>0\ ,\quad 0<6\le\psi(x(t))=6+\frac{x^2}{1+x^2}\le 7\ \ ,\ \ for\ \ all\ \ x\in R\ ,$$

$$\frac{g_1(t,x(t))}{g(x(t))}=\frac{x(t)\left(t^2+x^4(t)\right)}{x(t)}=t^2+x^4(t)\ge t^2=q(t)\qquad for\ \ x\ne 0\ \ and\ \ t\in[t_0,\infty)$$

$$If\ we\ take\ \ \rho(t)=\frac{1}{t}\ \ and\ \ H(t,s)=(t-s)^2\ , Then\quad \frac{\partial H(t,s)}{\partial s}=-2(t-s)\ ,\ \ h(t,s)=2\ ,\ \ t>0,$$

$$X(t,t_0)=\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)ds=\frac{1}{(t-t_0)^2}\int_{t_0}^{t}(t-s)^2 s ds=\frac{1}{(t-t_0)^2}\left[\frac{t^4}{12}-\frac{t^2t_0^2}{2}+\frac{2tt_0^3}{3}-\frac{t_0^4}{4}\right],$$

$$Y(t,t_0)=\frac{1}{H(t,t_0)}\int_{t_0}^{t}r(s)\rho(s)\left[\gamma(s)\sqrt{H(t,s)}+l_3h(t,s)\right]^2 ds\quad ;\ \gamma(s)=-\frac{l_3\dot{\rho}(s)}{\rho(s)}=-7.\frac{(-1)}{s}=\frac{7}{s}\ ,$$

$$Y(t,t_0)=\frac{1}{(t-t_0)^2}\int_{t_0}^{t}\left[\frac{7}{s}(t-s)+14\right]^2 ds=\frac{49}{(t-t_0)^2}\int_{t_0}^{t}\left[\frac{t}{s}+1\right]^2 ds$$

$$=\frac{49}{(t-t_0)^2}\left[2t\ln t+\frac{t^2}{t_0}-2t\ln t_0-t_0\right],$$

$$\limsup_{t\to\infty}\left[X(t,t_0)-\frac{1}{4c_1}Y(t,t_0)\right]$$

$$=\limsup_{t\to\infty}\left[\frac{1}{(t-t_0)^2}\left\langle\frac{t^4}{12}-\frac{t^2t_0^2}{2}+\frac{2tt_0^3}{3}-\frac{t_0^4}{4}-\frac{7}{2}t\ln t-\frac{7t^2}{4t_0}+\frac{7}{2}t\ln t_0+\frac{7t_0}{4}\right\rangle\right]=\infty,$$

It follows from theorem 2.3 that the given equation (16) is oscillatory.

Remark 2.3: Theorem 2.3 extends the results of A.Tiryaki and A.Zafar [14], S. R. Grace [5] and CH. G. Philos [15].

Theorem 2.4 : Suppose that (O_1) holds. And moreover , ρ , h *and* H be as in theorem 2.3 and

$$O_7:\ \ \limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}r(s)\rho(s)\left[h(t,s)-\frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds<\infty,$$

.

$$O_8: \ \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s)ds = \infty,$$

Then equation (1) is oscillatory.

Proof. Let $x(t)$ be a non oscillatory solution of equation (1) and assume that $x(t) > 0$ *for all* $t \geq T_1 \geq t_0$.

Define $$\omega(t) = \frac{\rho(t)r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}, \qquad t \geq T_2, \qquad (17)$$

Differentiating (17) and using (1), (O_1), we see that

$$\dot{\omega}(t) \leq -\rho(t)q(t) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - ll_1 \frac{1}{\rho(t)r(t)}\omega^2(t), \qquad t \geq T_2.$$

Then, for all $t \geq T_2$, we obtain

$$\int_{T_2}^{t} H(t,s)\rho(s)q(s)ds \leq -\int_{T_2}^{t} H(t,s)\dot{\omega}(s)ds + \int_{T_2}^{t} \frac{H(t,s)\dot{\rho}(s)}{\rho(s)}\omega(s)ds - k\int_{T_2}^{t} \frac{H(t,s)}{\rho(s)r(s)}\omega^2(s)ds,$$

Where $k = ll_1$ is a positive constant.

Then, for all $t \geq T_2$, we have

$$\int_{T_2}^{t} H(t,s)\rho(s)q(s)ds \leq -\left[H(t,s)\omega(s)\Big|_{T_2}^{t} - \int_{T_2}^{t} \frac{\partial H(t,s)}{\partial s}\omega(s)ds \right] + \int_{T_2}^{t} \frac{H(t,s)\dot{\rho}(s)}{\rho(s)}\omega(s)ds$$
$$- k\int_{T_2}^{t} \frac{H(t,s)}{\rho(s)r(s)}\omega^2(s)ds$$
$$\leq H(t,T_2)\omega(T_2) - \int_{T_2}^{t} \left[h(t,s)\sqrt{H(t,s)} + \frac{H(t,s)\dot{\rho}(s)}{\rho(s)} \right]\omega(s)ds - k\int_{T_2}^{t} \frac{H(t,s)}{\rho(s)r(s)}\omega^2(s)ds$$

$$\leq H(t,T_2)\omega(T_2) - \int_{T_2}^{t}\left\{\frac{k_3 H(t,s)}{\rho(s)r(s)}\omega^2(s) + \sqrt{H(t,s)}\left[h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]\omega(s)\right\}ds$$

$$\leq H(t,T_2)\omega(T_2) + \int_{T_2}^{t}\frac{\rho(s)r(s)}{4k}\left[h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\right]^2 ds$$

$$-\int_{T_2}^{t}\left\{\sqrt{\frac{k_3 H(t,s)}{\rho(s)r(s)}}\omega(s) + \frac{1}{2}\sqrt{\frac{\rho(s)r(s)}{k}}\left[h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]\right\}^2 ds$$

Then , for all $T_2 \geq t_0$, we have

$$\leq H(t,T_2)\omega(T_2) + \frac{1}{4k}\int_{T_2}^{t}\rho(s)r(s)\left[h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)}\sqrt{H(t,s)}\right]^2 ds \;,\quad \textit{for all } T_2 \geq t_0\,, \tag{18}$$

Now if we divide (18) by $H(t,t_0)$, take the upper limit as $t \to \infty$, and apply (O_7), we obtain

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\rho(s)q(s)ds < \infty.$$

This contradicts to the condition (O_8); hence, the proof is completed.

Example 2.4: Consider the equation

$$\left[\left(\frac{t^2+2}{t^2+3}\right)\left(\frac{x^4(t)+1}{x^4(t)+5}\right)\dot{x}(t)\right]^{\bullet} + x^5(t)\left[\frac{2}{t} + 2\sin t + x^4(t)\right] = 0,\quad t > 0. \tag{19}$$

We note that

$$r(t) = \frac{t^2+2}{t^2+3} > 0,\ \forall t \geq t_0 > 0,\ \ \psi(x) = \frac{x^4+1}{x^4+5} > 0 \quad \textit{and} \quad \frac{1}{\psi(x)} = \frac{x^4+5}{x^4+1} \geq 1\,,\ \forall x \in R,$$

$$\frac{g_1(t,x(t))}{g(x(t))} = \frac{x^5(t)\left[\frac{2}{t} + 2\sin t + x^4(t)\right]}{x^5(t)} = \frac{2}{t} + 2\sin t + x^4(t)$$

.

$$\geq \frac{2}{t} + 2\sin t = q(t), \; for \;\; all \;\; x \neq 0 \;\; and \;\; t \in [t_0, \infty),$$

$$Let \;\; H(t,s) = (t-s)^2 \geq 0 \,, \;\; \forall t \geq s \geq t_0 > 0 \;,$$

$$Then \;\; \frac{\partial H(t,s)}{\partial s} = -2(t-s) \qquad and \;\; then \qquad h(t,s) = 2$$

$$and \;\; taking \quad \rho(t) = 3 > 0 \qquad for \;\; t > 0 \,, \qquad then \quad \dot{\rho}(t) = 0$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \rho(s) r(s) \left[h(t,s) - \frac{\dot{\rho}(s)}{\rho(s)} \sqrt{H(t,s)} \right]^2 ds = \limsup_{t\to\infty} \frac{12}{(t-t_0)^2} \int_{t_0}^{t} \frac{s^2+2}{s^2+3} ds$$

$$= \limsup_{t\to\infty} \frac{12}{(t-t_0)^2} \int_{t_0}^{t} \left[1 - \frac{1}{s^2+3} \right] ds < \infty,$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) \rho(s) q(s) ds = \limsup_{t\to\infty} \frac{3}{(t-t_0)^2} \int_{t_0}^{t} (t-s)^2 \left[\frac{2}{s} + 2\sin s \right] ds = \infty.$$

It follows from Theorem 2.4 that the given equation (19) is oscillatory.

Remark 2.4 Theorem 2.4 extends the results of Grace [6] , [7] , Ohriska and A.Zulova [12] and A.Tiryaki and A.Zafar [14].

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