Sirte University Scientific Journal Vol. 13, No1 (2023) 55-59

Sirte University Scientific Journal (SUSJ)

Journal home page: http://journal.su.edu.ly/index.php/SUSJ/index

DOI: 10.37375/ISSN:2518-5454

Subordination Results of Analytic Functions Defined by Convolution

A. A. Hussain[1], M. K. Aouf[2], and A. O. Mostafa[3]

[1]*Department of Mathematics, Faculty of Education, Sirte University, Libya.*
[2,3]*Department of Mathematics, Faculty of Science, Mansura University, Egypt.*

© SUSJ2023.

DOI: https://doi.org/10.37375/sjfssu.v13i1.1373

ARTICLE INFO:

Received 20 June 2022.
Accepted 14 February 2023.
Available online 01 June 2023

Keywords: *Analytic function, univalent functions, Hadamard product, subordinating, factor sequence*

A B S T R A C T

The purpose of this paper is to study some known interesting subordination results for analytic functions defined in the open unite disk $U = \{z \in \mathbb{C}: |z| < 1\}$. Especially, it is to obtain subordination results for a family of univalent functions which are defined by means of the convolution. Relevant connections of the results presented here with those obtained in earlier works are also pointed out. However, our results generalize and extend some earlier results in the literature.

1 Introduction

In this work, we prove several subordination relationships involving the functions in subclass $\Psi(\Omega, \Upsilon, \delta, \lambda)$. In our proposed investigation of functions in the subclass of the normalized analytic function class S, we need the following definitions and result.

The class of analytic functions in the open unit disc $U = \{z \in \mathbb{C}: |z| < 1\}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

is denoted by S For the function $f(z)$ and $g(z) \in S$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (1.2)$$

the Hadamard product (or convolution) is given by

$$(f * g)(z) = \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \qquad (1.3)$$

For two functions $f(z)$ and $\emptyset(z)$ analytic in , say that the function $f(z)$ and $\emptyset(z)$ is subordinate to $\emptyset(z)$ in U written $f(z) \prec \emptyset(z)$, if there exists a Schwarz function $w(z)$ which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = \emptyset(w(z))$. Indeed it is known that

$f(z) \prec \emptyset(z) \Longrightarrow f(0) = \emptyset(0)$ *and* $f(0) \subset \emptyset(0)$.

Furthermore, if the function $\emptyset(0)$ univalent in U, then we have the following equivalence (see [2] and [5]);

$$f(z) \prec \emptyset(z) \Leftrightarrow f(0) = \emptyset(0) \text{ and } f(U) \subset \emptyset(U).$$

Definition: *(Subordinating Factor Sequence) [8]. A sequence $\{c_n\}_{n-1}^{\infty}$ of complex numbers is said to be a subordination factor sequence if, whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in U, we have*

$$\sum_{n=1}^{\infty} c_n a_n z^n \prec f(z)(z \in U; a_1 = 1), \qquad (1.4)$$

Corresponding author: **E-mail:** *aisha84_hussain@yahoo.com*

for $\Omega(z) = z + \sum_{n=2}^{\infty} \gamma_n z^n \ \ and \ \ \Upsilon(z) = z + \sum_{n=2}^{\infty} \upsilon_n z^n \ in\ U, where\ \gamma_n \geq \upsilon_n \ > 0 (n \geq 2).$

Murugusundaramoorthy and Frasin [6], defined the subclass $\Psi(\Omega, \Upsilon, \delta, \lambda)$ of Ψ consisting of functions $f(z)$ of the form (1.1) and satisfying

$$R\left\{\frac{(f*\Omega)(z)}{(1-\delta)(f*\Upsilon)(z)+\delta(f*\Omega)(z)}\right\} \geq \lambda \ (z \in U), \quad (1.5)$$

where $0 \leq \delta < 1$ and $0 \leq \lambda < 1$.

We note that:

(i) $\Psi\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}, 0, \lambda\right) = \Psi^*(\lambda)$ (the class of starlike functions of order λ and $\Psi\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, 0, \lambda\right) = K(\lambda)$ (the class of convex functions of order λ (see Robretson [7]), where $\Psi^*(\lambda)$ $=\Psi^*(\lambda)$ and $K(\lambda) = K(\lambda)$

(ii) $\Psi(zg'(z), g(z), \delta, \lambda) = \Psi(g(z), \delta, \lambda)$(see Aouf et al. [1, with = 0]), where g(z) is given by (1.2).

For another choices of $\Omega(z)$ and $\Upsilon(z)$ we have the following new subclasses:

(i) Putting $\Omega(z) = z + \sum_{n=2}^{\infty}\left(\frac{1+l+\mu(n-1)}{1+l}\right)^m z^n$ and $\Upsilon(z) = z + \sum_{n=2}^{\infty}\Gamma_n(\alpha_1) z^n$, where

$$\Gamma_n(\alpha_1) = \frac{(\alpha_1)_{n-1} \ldots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_s)_{n-1}} . \frac{1}{(n-1)!} \qquad (1.6)$$

$(\alpha_i (i = 1,2, \ldots q);\ \beta_j \{-1, -2, \ldots\}, (j = 1,2, \ldots, s)$

Are real and $\beta_j \neq \{0, -1, \ldots\};\ \mu, l \geq 0, m \in N_0 = N \cup \{0\}, N = \{1,2, \ldots\})$,

and

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & k = 0 \\ a(a+1)(a+2) \ldots (a+k-1); & k \in \mathrm{N} \end{cases}$$

in (1.5) the subclass $\Psi(\Omega, \Upsilon, \delta, \lambda)$ reduces to the subclass $\Psi_{q,s}(\mathrm{m}, [\alpha_1], \delta, \lambda)$

$$= \left\{ f(z) \in \Psi : \mathrm{Re}\left\{\frac{I^m(\mu,l)(f)(z)}{(1-\delta)H_{q,s}(\alpha_1)(f)(z) + \delta I^m(\mu,l)(f)(z)}\right\} \geq \lambda \right\},$$

where the operators $I^m(\mu, l)$ and $H_{q,s}(\alpha_1)$ respectively, were introduced and studied by Cãtas et al. [3] and Dziok-Srivastava [4], respectively, which generaliz of many other operators.

(ii) Putting $\Omega(z) = z + \sum_{n=2}^{\infty}\Gamma_n(\alpha_1) z^n$ and $\Upsilon(z) = z + \sum_{n=2}^{\infty}\left(\frac{1+l+\mu(n-1)}{1+l}\right)^m z^n$, in (1.5) the subclass $\Psi(\Omega, \Upsilon, \delta, \lambda)$ reduces to the subclass $\Psi^*([\alpha_1], m, \delta, \lambda)$

$$= \left\{ f(z) \in \Psi : \mathrm{Re}\left\{\frac{H_{q,s}(\alpha_1)(f)(z)}{(1-\delta)I^m(\mu,l)(f)(z) + \delta H_{q,s}(\alpha_1)(f)(z)}\right\} \geq \lambda \right\}.$$

2 Main result

In the reminder of this paper, we assume that:

$0 \leq \delta < 1, 0 \leq \lambda < 1, \gamma_n \geq \upsilon_n > 0 (n \geq 2)\ and\ z \in U$.

To prove our result, we need to the following lemmas.

Lemma 1 *[8]. The sequence* $\{c_n\}_{n=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$Re\left\{1 + 2\sum_{n=1}^{\infty} c_n z^n\right\} > 0. \quad (2.1)$$

Lemma 2 *[6]. If* $f(z) \in S$, *satisfies*

$$\sum_{n=1}^{\infty}\{\gamma_n - [\upsilon_n + \delta(\gamma_n - \upsilon_n)\lambda]\}|a_n| \leq 1 - \lambda, \qquad (2.2)$$

then $f(z) \in \Psi(\Omega, \Upsilon, \delta, \lambda)$.

Corollary 1. *If* $f(z) \in \Psi(\Omega, \Upsilon, \delta, \lambda)$, *then*

$$|a_n| \leq \frac{1-\lambda}{\{\gamma_n - [\upsilon_n + \delta(\gamma_n - \upsilon_n)\lambda]\}} (n \geq 2). \qquad (2.3)$$

The equality holds for

$$f(z) = z + \frac{1-\lambda}{\{\gamma_n - [\upsilon_n + \delta(\gamma_n - \upsilon_n)\lambda]\}} z^n. \qquad (2.4)$$

Let $\Psi^*(\Omega, \Upsilon, \delta, \lambda)$ denote the class of $f(z) \in S$ whose coefficients satisfy the condition (2.2). We note that $\Psi^*(\Omega, \Upsilon, \delta, \lambda) \subseteq \Psi(\Omega, \Upsilon, \delta, \lambda)$.

Open Access Article is distributed under a CC BY 4.0 Licence

Theorem 1. *If* $f(z) \in \Psi^*(\Omega, \Upsilon, \delta, \lambda)$. *Then*

$$\frac{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda}{2\{1 - \lambda + (\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}}(f * h)(z) \prec h(z), \quad (2.5)$$

for every function $h(z) \in \mathrm{K}$ *, and*

$$Re\{f(z)\} > -\frac{\{1 - \lambda + (\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}}{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda}. \quad (2.6)$$

The constant factor $\frac{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda}{2\{1-\lambda+(\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}}$ *in the subordination result (2.5) can not be replaced by a large one.*

Proof. Let $f(z) \in \Psi^*(\Omega, \Upsilon, \delta, \lambda)$. and suppose that

$h(z) = z + \sum_{n=2}^{\infty} c_n z^n$,
then

$$\frac{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda}{2\{1 - \lambda + (\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}}(f * h)(z)$$

$$= \frac{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda}{2\{1 - \lambda + (\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}}\left(z + \sum_{n=2}^{\infty} a_n c_n z^n,\right). \quad (2.7)$$

Thus, by using Definition, the subordination result holds true if

$$\left\{\frac{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda}{2\{1 - \lambda + (\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}} a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view Lemma 1 , this is equivalent to the following inequality:

$$Re\left\{1 + \sum_{n=2}^{\infty} \frac{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda}{\{1 - \lambda + (\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}} a_n z^n,\right\} > 0. \quad (2.8)$$

Now, since

$$\Delta(n) = \{\gamma_n - [\nu_n + \delta(\gamma_n - \nu_n)]\lambda\},$$

is an increasing function of $n (n \geq 2)$, we have

$$Re\left\{1 + \sum_{n=2}^{\infty} \frac{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda}{\{1 - \lambda + (\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}} a_n z^n\right\}$$

$$= Re\left\{1 + \frac{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda}{\{1 - \lambda + (\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}} z + \sum_{n=2}^{\infty}\{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda\} a_n z^n\right\}$$

$$\geq 1 - \frac{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda}{\{1 - \lambda + (\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}} r - \frac{1}{\{1 - \lambda + (\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}} \sum_{n=2}^{\infty}\{\gamma_n - [\nu_n + \delta(\gamma_n - \nu_n)]\lambda\} a_n r^n$$

$$\geq 1 - \frac{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda}{\{1 - \lambda + (\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}} r - \frac{1 - \lambda}{\{1 - \lambda + (\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}} r$$

$$> 0 (|z| = r < 1),$$

where we have also made use of assertion (2.2) of Lemma. Thus (2.8) holds true in· This proves the inequality (2.5). The inequality (2.6) follows from (2.5) by taking the convex function

$$h(z) = \frac{z}{1 - z} = z + \sum_{n=2}^{\infty} z^n \in \mathrm{K}.$$

To prove the sharpness of the constant

$$\frac{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda}{2\{1 - \lambda + (\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}}$$

we consider the function $f_0(z) \in \Psi^*(\Omega, \Upsilon, \delta, \lambda)$ given by

$$f_0(z) = z - \frac{1 - \lambda}{\{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda\}} z^2. \quad (2.9)$$

Thus from (2.5), we have

$$\frac{\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)]\lambda}{2\{1 - \lambda + (\gamma_2 - [\nu_2 + \delta(\gamma_2 - \nu_2)\lambda])\}} f_0(z) \prec \frac{z}{1 - z}.$$

It is easily verified that

Open Access Article is distributed under a CC BY 4.0 Licence

$$\min_{|z|\le r}\left\{Re\left(\frac{\gamma_2-[\nu_2+\delta(\gamma_2-\nu_2)]\lambda}{2\{1-\lambda+(\gamma_2-[\nu_2+\delta(\gamma_2-\nu_2)\lambda])\}}f_0(z)\right)\right\} = \frac{-1}{2} \qquad (2.10)$$

This show that the constant

$$\frac{\gamma_2-[\nu_2+\delta(\gamma_2-\nu_2)]\lambda}{2\{1-\lambda+(\gamma_2-[\nu_2+\delta(\gamma_2-\nu_2)\lambda])\}}$$

is the best possible. This completes the proof of Theorem.

Putting $\Omega(z) = z + \sum_{n=2}^{\infty}\left(\frac{1+l+\mu(n-1)}{1+l}\right)^m z^n$ $\left(or\ \gamma_n = \left(\frac{1+l+\mu(n-1)}{1+l}\right)^m\right)$ and $\Upsilon(z) = z + \sum_{n=2}^{\infty}\Gamma_n(\alpha_1)z^n$ $or(\nu_n = \Gamma_n(\alpha_1))$

where $\Gamma_n(\alpha_1)$ is defined by (1.6), in Lemma 2 and Theorem 1,

■

Now, we obtain the following corollary:

Corollary 2: *Let the function* $f(z) \in \Psi_{q,s}(\mathrm{m},[\alpha_1],\delta,\lambda)$ *and satisfy*

$$\sum_{n=2}^{\infty}\left\{\left(\frac{1+l+\mu(n-1)}{1+l}\right)^m - \left[\Gamma_n(\alpha_1) + \delta\left(\left(\frac{1+l+\mu(n-1)}{1+l}\right)^m - \Gamma_n(\alpha_1)\right)\right]\lambda\right\}|a_n| \le 1-\lambda. \qquad (2.11)$$

Then for every function $h(z) \in \mathrm{K}$ *we have*

$$\frac{(1+l+\mu)^m-[\Gamma_2(\alpha_1)(1+l)^m+\delta((1+l+\mu)^m-\Gamma_2(\alpha_1)(1+l)^m)]\lambda}{2\{(1-\lambda)(1+l)^m+(1+l+\mu)^m-[\Gamma_2(\alpha_1)(1+l)^m+\delta((1+l+\mu)^m-\Gamma_2(\alpha_1)(1+l)^m)]\lambda\}}$$

$$(f*h)(z) \prec h(z), \qquad (2.12)$$

and

$$Re\{f(z)\} > -\frac{(1-\lambda)(1+l)^m+(1+l+\mu)^m-[\Gamma_2(\alpha_1)(1+l)^m+\delta((1+l+\mu)^m-\Gamma_2(\alpha_1)(1+l)^m)]\lambda}{(1+l+\mu)^m-[\Gamma_2(\alpha_1)(1+l)^m+\delta((1+l+\mu)^m-\Gamma_2(\alpha_1)(1+l)^m)]\lambda} \qquad (2.13)$$

The constant factor

$$\frac{(1+l+\mu)^m-[\Gamma_2(\alpha_1)(1+l)^m+\delta((1+l+\mu)^m-\Gamma_2(\alpha_1)(1+l)^m)]\lambda}{2\{(1-\lambda)(1+l)^m+(1+l+\mu)^m-[\Gamma_2(\alpha_1)(1+l)^m+\delta((1+l+\mu)^m-\Gamma_2(\alpha_1)(1+l)^m)]\lambda\}}$$

in the subordination result (2.12) cannot be replaced by a large one.

Putting $\Omega(z) = z + \sum_{n=2}^{\infty}\Gamma_n(\alpha_1)z^n$ $or(\gamma_n = \Gamma_n(\alpha_1))$ and $\Upsilon(z) = z + \sum_{n=2}^{\infty}\left(\frac{1+l+\mu(n-1)}{1+l}\right)^m z^n$ $\left(or\ \nu_n = \left(\frac{1+l+\mu(n-1)}{1+l}\right)^m\right)$, in Lemma 2 and Theorem 1, we obtain the following corollary:

Corollary3. *Let the function* $f(z) \in \Psi^*_{q,s}([\alpha_1],\mathrm{m},\delta,\lambda)$ *and satisfy*

$$\sum_{n=2}^{\infty}\left\{\Gamma_n(\alpha_1 - \left[\left(\frac{1+l+\mu(n-1)}{1+l}\right)^m + \delta\left(\Gamma_n(\alpha_1 - \left(\frac{1+l+\mu(n-1)}{1+l}\right)^m\right)\right]\lambda\right\}|a_n| \le 1-\lambda. \qquad (2.14)$$

Then for every function $h(z) \in \mathrm{K}$ *we have*

$$\frac{\Gamma_2(\alpha_1)(1+l)^m-[(1+l+\mu)^m+\delta((1+l)^m\Gamma_2(\alpha_1)-(1+l+\mu)^m)]\lambda}{2\{(1-\lambda)(1+l)^m+\Gamma_2(\alpha_1)(1+l)^m-[(1+l+\mu)^m+\delta((1+l)^m\Gamma_2(\alpha_1)-(1+l+\mu)^m)]\lambda\}}$$

$$(f*h)(z) \prec h(z), \qquad (2.15)$$

and

$$Re\{f(z)\} > -\frac{(1-\lambda)(1+l)^m+\Gamma_2(\alpha_1)(1+l)^m-[(1+l+\mu)^m+\delta((1+l)^m\Gamma_2(\alpha_1)-(1+l+\mu)^m)]\lambda}{\Gamma_2(\alpha_1)(1+l)^m-[(1+l+\mu)^m+\delta((1+l)^m\Gamma_2(\alpha_1)-(1+l+\mu)^m)]\lambda}. \qquad (2.16)$$

The constant factor

$$\frac{\Gamma_2(\alpha_1)(1+l)^m-[(1+l+\mu)^m+\delta((1+l)^m\Gamma_2(\alpha_1)-(1+l+\mu)^m)]\lambda}{2\{(1-\lambda)(1+l)^m+\Gamma_2(\alpha_1)(1+l)^m-[(1+l+\mu)^m+\delta((1+l)^m\Gamma_2(\alpha_1)-(1+l+\mu)^m)]\lambda\}}$$

in the subordination result (2.15) cannot be replaced by a large one.

Remark. Putting $\boldsymbol{\gamma_n = g(z)}$ and $\boldsymbol{\Omega(z) = zg'(z)}$ where $\boldsymbol{g(z)}$ is given by (1.2), in Lemma 2 and Theorem 1, respectively, we obtain the results obtained by Aouf et al. [1, Lemma 2 and Theorem 1, respectively, with $\boldsymbol{\beta = 0}$].

3 Conclusions

In this recent paper, we considered some analytic functions defined by convolution. By selecting different values for both the function $\boldsymbol{g(z)}$ and the parameters $\boldsymbol{\delta, \lambda}$, we came out with some new subordination results which are interesting in their own right.

4 Acknowledgements

The authors would like to thank the referee for his helpful comments and suggestion.

References

M. K. Aouf, A. A. Shamandy, A. O. Mostafa and A. K. Wagdy, Subordination results for subclasses of analytic functions defined by convolution, Acta Univ. Apulensis, 31(2012), 129-140.

T.Bulboaca, Differential Subordination and Subordinations, Recent Results, House of-Scientific Book Publ..Cluj-Napoca. 2005.

Open Access Article is distributed under a CC BY 4.0 Licence

A. Cãtãs, G. I.Oros and G. Oros, Differential subordinations associated with multiplier transformations, Abstract Appl. Anal., 2008(2008), ID845724, 1-11.

J.D ziok and H. Srivastava, Classes of analytic functions with the generalized hypergeometric function, Appl. Math. Comput., 103(1999),1-13.

S.S. Miller and P. T. Mocanu, Differential Subordinations Theory and Applications, in: Series on Monograph and Textbooks in Pure and Appl. Math., 255, Marcel Dekker, Inc, New York, 2000.

G. Murugusundaramoorthy and B. A. Frasin, Family of analytic functions defined by convolution, Romai J., 1(2011), no. 7, 125-139.

M. S. Roberston, On the theory of univalent functions, Ann. Math., 37(1936), 374-408.

H. S. Wilf, Subordinatinting factor sequence for convex maps of the unit circle, Proc.

Open Access Article is distributed under a CC BY 4.0 Licence