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On Some Numerical Methods for Solving Fredholm Integral Equations with Continuous Kernel

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Abstract

Fredholm integral equations with continues kernels arise for instance in the boundary integral equations either directly or as a result of treating different kinds of singular kernels using some regularization techniques. Here, we show a comparison of the convergence of two well-known numerical methods for solving integral equations. These methods are the collocation method and the Galerkin method. An illustrated examples for second kind Fredholm integral equations of continuous kernel show that the collocation method seems to converge faster than the Galerkin method.

Keywords: *Fredholm integral equation, Collocation method, Galerkin method, Boundary integral equations.*

1. Introduction

The application of integral equations with different kinds of kernels is an important subject within applied sciences. Integral equations are used as mathematical models for many physical situations. Also, the rapid development of computer engineering has around the considerable interest of researchers for the development of numerical methods for the solution of applied problems. Many different methods can be used for solving the integral equations analytically. These methods are: Fourier transformation methods [1]; singular integral method, Cauchy method [2]; orthogonal polynomials method [3, 4]; degenerate kernel method [5, 6], Potential theory method [7]; and Kreins method [8]. For a rigorous insight into the analytic methods for solving the integral equations we refer the reader to the references [9, 10].

The numerical methods take an important place in solving integral equations when the kernel has either continuous or discontinuous form. For some of these numerical methods, in the linear and nonlinear integral equations we state projection-iteration methods [11, 12]; collocation method [13, 14, 15]; Projection method [16, 17], Galerkin method [18, 19]. More information for the numerical methods can be found in Atkinson [20, 21], Delves and Mohamed [22], Baker [23] and Goldberg [24].

The Fredholm Integral Equations (FIEs) of continuous (non-singular) kernel arise in many applications. For example, one obtains such equations as a result of implementing the boundary element method [25] which reduces the differential equation to an integral equation. Therefore, one obtains boundary integral equations of continuous kernels either directly or as a result of treating different kinds of singular kernels using some regularization techniques [26, 27, 28].
Conceptually, solving the integral equation depend on its kernel. Thus if the kernel is continuous and can be written as a multiplication of two functions, one can use the degenerate kernel method [22, 23, 24]. On the other hand, if the kernel cannot be written as multiplication of two functions, one can use the approximate kernel method, or the iterated method. These methods, for example, are square method, block by block method, Runge–Kutta method, collocation method, Galerkin method and Adomian method [22, 23, 24].

In this paper, we discuss the solution of a second kind FIEs with continuous kernel using some numerical methods such as the collocation and Galerkin methods. In section one; we set up the notation for the kind of the integral equation that we are using in this paper. Then in section two and three we briefly present the core idea of both the collocation and the Galerkin methods, respectively. Also for the sake of comparison, some examples in the linear case are solved by each method. Finlay, a discussion and conclusion is drawn on the performance of both the collocation and the Galerkin methods.

2. Types of integral equations

Equations in which the unknown function appears under an integral sign and may be add to one of both sides are called integral equations. The integral equation takes the form [1],

$$\mu\,\varphi(x)-\lambda\int_a^b K(x,y)\varphi(y)\,dy=f(x)\,. \tag{1}$$

Where $f(x)$, and $K(x, y)$ are known functions, which are named the free term and the kernel of the integral equation, respectively. The numerical coefficient λ is called the parameter of the integral equation and $\varphi(x)$, is the unknown function to be determined.
There are numbers of classification of linear integral equations that distinguish different kinds of equations. We take $\mu = 0$, for the FIE of the first kind, while μ =constant $\neq 0$ for the FIE of the second kind, and $\mu = \mu(x)$, for the third kind. After giving a brief definition of the type of equation that we consider in this paper, next we will present two numerical methods and illustrated examples solved by both methods.

3. The Collocation method

Here we show how to apply the collocation method [13, 14]; to solve the following FIE of the second kind,

$$\varphi(x) = f(x) + \int_a^b K(x,y)\varphi(y)\, dy \, . \tag{2}$$

We suppose that the approximate solution of the FIE (2) is expressed as,

$$S_n(x) = \sum_{k=1}^{n} C_k\, \varphi_k(x). \tag{3}$$

Such that the functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are linearly independent. Implementing the approximate solution (3) in the FIE (2) leads to

$$S_n(x) = f(x) + \int_a^b K(x,y)\, S_n(y)\, dy + E(x, C_1, C_2, ..., C_n), \tag{4}$$

where $E(x, C_1, C_2, \ldots, C_n)$ is the computation error. Then substitute the points $x_1, x_2, \ldots, x_n$ in equation (4) to obtain

$$S_n(x_i) = f(x_i) + \int_a^b K(x_i, y) S_n(y) dy, x_i \in [a,b] \; , \; i = 1, 2, \ldots, n. \tag{5}$$

To find the constants $C_1, C_2, \ldots, C_n$, of the approximate solution $S_n(x)$ given in equation (3), we use the values of the functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ and calculate the integrals . Then substitute by the points $x_1, x_2, \ldots, x_n$, such that $E(x_i, C_1, C_2, \ldots, C_n) = 0$, we obtain n algebraic linear equations in n unknowns $(C_1, C_2, \ldots, C_n)$. A final step is to solve the resulting linear system and obtain the approximate solution $S_n(x)$ defined by equation (3). Next we give some demonstration examples showing the method.

Example 1: Solve the following integral equation using the collocation method

$$\varphi(x) + 2\int_0^1 e^{x-y} \varphi(y) dy = 2x\, e^x, \tag{6}$$

where the exact solution is given by the function

$$\varphi(x) = e^x \left(2x - \frac{2}{3}\right).$$

Solution: Suppose that the approximate solution takes the form

$$S_n(x) = \sum_{k=1}^{n} C_k \varphi_k(x).$$

.

Where we take,

$$n = 5 \quad , \varphi_1(x) = 1 \ , \ \varphi_2(x) = x \ , \ \varphi_3(x) = x^2 , \ \varphi_4(x) = x^3 , \ \varphi_5(x) = x^4 \ .$$

That is, we have

$$S_n(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4$$

Substituting this approximate solution in the FIE (6) leads to,

$$S_n(x) + 2\int_0^1 e^{x-y} S_n(y) dy = 2x \ e^x + E(x, C_1, C_2, C_3, C_4, C_5) \ .$$

That is

$$\left(C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4\right) + 2\int_0^1 e^{x-y}\left(C_1 + C_2 y + C_3 y^2 + C_4 y^3 + C_5 y^4\right) dy =$$

$$2xe^x + E(x, C_1, C_2, C_3, C_4, C_5) \ ,$$

Such that the last term represents the error. Thus we obtain the following linear system

$$2.264241118C_1 + 0.22785788C_4 + 0.17567264C_5 + 0.528482235C_2 + 0.321205588C_3 = 0.$$

$$2.623317728C_1 + 0.9285846224C_2 + 0.4749361390C_3 + 0.3082003120C_4 + 0.2294743848C_5 = 0.6420127085$$

$$3.084381222C_1 + 1.371319903C_2 + 0.7795784852C_3 + 0.5006741368C_4 + 0.3521352184C_5 = 1.648721271$$

$$3.676398468C_1 + 1.868796901C_2 + 1.242492235C_3 + 0.9042501400C_4 + 0.6883052318C_5 = 3.175500026$$

$$4.436563656C_1 + 2.436563656C_2 + 1.873127313C_3 + 1.619381940C_4 + 1.477527745C_5 = 5.436563656$$

By using Maple18 we solve this system, and we get

$$C_1 = -0.666648983700731, \ C_2 = 1.32113354685743, C_3 = 1.76621887428895,$$
$$C_4 = 0.622365994141024, C_5 = 0.581354401742185.$$

So, we obtain the following approximate solution of the FIE (6),

$$S_5(x) = -0.666648983700731 + 1.32113354685743x + 1.76621887428895x^2 +$$
$$0.622365994141024\, x^3 + 0.581354401742185x^4.$$

Table 1 shows a comparison of the exact (analytical) solution $\varphi(x)$ of the FIE (6) against the numerical solution $S_n(x)$ for different values of x taken in the given interval [0, 1]. The error can be determined as the absolute value of the difference between the exact solution and numerical solution as shown in the fourth column of Table 1. We notice that by increasing the number of points to n = 7, we obtain more accurate solution of the FIE (6) as shown in the fourth column of Table 2.

Table 1: The exact and numerical solutions of the FIE (6) using collocation method for n = 5.

x	$\varphi(x)$ Analytical solution	$S_5(x)$ Numerical solution	$\lvert\varphi(x)-S_5(x)\rvert$ Error
0	-0.6666666667	-0.666648983700731	0.0000176829659356281
0.25	-0.2140042362	-0.213981533053056	0.0000227031469440253
0.5	0.5495737569	0.549602907676735	0.0000291507767345989
0.75	1.764166681	1.76420411318436	0.0000374321843561454
1	3.624375770	3.62442383332885	0.0000480633288546528

Table 2: The exact and numerical solutions of the FIE (6) using collocation method for n = 7.

x	$\varphi(x)$ Analytical solution	$S_7(x)$ Numerical solution	$\lvert\varphi(x)-S_7(x)\rvert$ Error
0	-0.6666666667	-0.666666609273182	$5.73934844094737\times10^{-8}$
0.25	-0.2140042362	-0.214003819445924	$4.16754076221215\times10^{-7}$
0.5	0.5495737569	0.549573828162462	$7.12624619314539\times10^{-8}$
0.75	1.764166681	1.76416638538647	$2.95613527656613\times10^{-7}$
1	3.624375770	3.62437588929795	$1.19297951517439\times10^{-7}$

Example 2: Use the collocation method to solve the following integral equation,

.

$$\phi(x)-\int_0^{\frac{\pi}{2}} x^5 \sin(xy)\,\phi(y)\,dy = 3x^2-2x^2\cos\left(\frac{\pi}{2}x\right)+\frac{1}{4}\pi^2x^4\cos\left(\frac{\pi}{2}x\right)-\pi x^3\sin\left(\frac{\pi}{2}x\right), \qquad (7)$$

where the exact solution is given as $\varphi(x) = x^2$.

Solution: Suppose that the approximate solution takes the form

$$S_n(x)=\sum_{k=1}^{n} C_k\,\varphi_k(x)\ .$$

We take,

$$n=5\ ,\ \varphi_1(x)=1\ ,\ \varphi_2(x)=x\ ,\ \varphi_3(x)=x^2,\ \varphi_4(x)=x^3,\ \varphi_5(x)=x^4\ .$$

Substituting this approximate solution in the FIE (7) leads to,

$$S_n(x)-\int_0^{\frac{\pi}{2}} x^5\sin(xy)\,S_n(y)\,dy = 3x^2-2x^2\cos\left(\frac{\pi}{2}x\right)+\frac{1}{4}\pi^2x^4\cos\left(\frac{\pi}{2}x\right)-\pi x^3\sin\left(\frac{\pi}{2}x\right)+E\left(x,C_1,\ldots,C_5\right)$$

To find the unknowns $(C_1,C_2,\ldots,C_5)$ in $S_n(x)$, we substitute by the points $x_1, x_2, x_3\ x_4, x_5$, taken in the interval $[0, \pi/2]$ such that the error is zero. Hence For $x = 0$, one has $C_1 = 0$.

For $x = 0.39269$, one has

$0.99561\ C_1 + 0.38813\ C_2 + 0.14886\ C_3 - 0.05385\ C_4 - 0.01502\ C_5 = 0.14886$.

For $x = 0.78539$, one has

$0.74534\ C_1 + 0.52587\ C_2 + 0.31667\ C_3 + 0.11221 C_4 - 0.10193\ C_5 = 0.31667$.

For $x = 1.17809$, one has

$-1.45819 C_1 - 1.22892\ C_2 - 1.31929 C_3 - 1.65353 C_4 - 2.26882\ C_5 = -1.31929$.

For $x = 1.570796327$, one has

$-9.84413\ C_1 - 8.31953\ C_2 - 8.07895 C_3 - 8.41708 C_4 - 9.08605\ C_5 = -8.07895$.

Using Maple 18, we solve this linear system for the unknowns C_1, C_2, C_3, C_4 , C_5 to obtain the values,

$C_1 = -3.41245\times10^{-20}$, $C_2 = 3.32217\times10^{-9}$, $C_3 = 0.99999$, $C_4 = 1.47975\times10^{-8}$, $C_5 = -4.71064\times10^{-9}$.

So, we obtain the following approximate solution of the FIE (7),

$$S_5(x) = -\ 3.41245\times10^{-20}\ -3.32217\times10^{-9}x\ +\ 0.99999x^2\ -1.47975\times10^{-8}\ x^3$$
$$-\ 4.71064\times10^{-9}x^4.$$

Table 3 shows a comparison of the exact (analytical) solution $\varphi(x)$ of the FIE (7) against the numerical solution $S_n(x)$ for different values of x taken in the given interval $[0, \pi/2]$. The error can be determined as the absolute value of the difference between the exact solution and numerical solution as shown in the fourth column of Table 3.

Table 3: The exact and numerical solutions of the FIE (7) using collocation method.

x	$\varphi(x)$ Analytical solution	$S_5(x)$ Numerical solution	$\lvert\varphi(x)-S_5(x)\rvert$ Error
0	0	-3.4124528×10^{-20}	3.4124528×10^{-20}
$\pi/14$	0.050355124	0.050355124	2.1975532×10^{-10}
$\pi/7$	0.201420498	0.201420497	1.2595044×10^{-10}
$3\pi/14$	0.453196120	0.453196120	3.3556864×10^{-10}
$2\pi/7$	0.805681992	0.805681991	2.3108902×10^{-10}
$5\pi/14$	1.258878113	1.258878114	3.8478842×10^{-10}
$3\pi/7$	1.812784482	1.812784483	4.3573545×10^{-10}
$\pi/2$	2.467401101	2.467401100	3.3528462×10^{-10}

Example 3: Solve the following integral equation using the collocation method

$$\varphi(x)-\int_0^1 (x+1)^3 e^{xy}\varphi(y)\,dy = e^x - (x+1)^2\left(e^{x+1}-1\right), \qquad (8)$$

where the exact solution is given by the function
$\varphi(x)= e^x$.

Solution: Suppose that the approximate solution takes the form

$$S_n(x)=\sum_{k=1}^{n} C_k\varphi_k(x).$$

Where we take,

$$n=5 \;,\; \varphi_1(x)=1\,,\; \varphi_2(x)=x\,,\; \varphi_3(x)=x^2,\; \varphi_4(x)=x^3,\; \varphi_5(x)=x^4\;.$$

That is, we have

$$S_n(x)=C_1+C_2x+C_3x^2+C_4x^3+C_5x^4$$

.

Substituting this approximate solution in the FIE (8) leads to,

$$S_n(x)+2\int_0^1 e^{x-y}S_n(y)dy = 2x\ e^x + E(x, C_1, C_2, C_3, C_4, C_5).$$

That is,

$$\left(C_1+C_2x+C_3x^2+C_4x^3+C_5x^4\right)-\int_0^1 (x+1)^3 e^{xy}\left(C_1+C_2y+C_3y^2+C_4y^3+C_5y^4\right)dy =$$

$$e^x-(x+1)^2\left(e^{x+1}-1\right)+E(x, C_1, C_2, C_3, C_4, C_5),$$

Such that the last term represents the error. Thus we obtain the following linear system

$$-0.399824918C_1-0.6115757236C_2-0.46831018C_3-0.3595163C_4-0.2899C_5=-1.319869971$$

$$-1.748717812C_1-1.123672626C_2-0.8857324526C_3-0.7209458219C_4-0.5994269362C_5=-3.465331018$$

$$-3.964622878C_1-2.158717643C_2-1.582922372C_3-1.284807900C_4-1.089741628C_5=-7.183554097$$

$$-7.446931425C_1-3.988612652C_2-2.769060075C_3-2.153452421C_4-1.785035352C_5=-13.26835612$$

$$-12.74625463C_1-7.000000002C_2-4.746254626C_3-3.507490746C_4-2.716291600C_5=-22.83794257$$

By using Maple18 we solve this system, and we get

$C_1 = 1.00029591104090,\ C_2 = 0.996100458779729, C_3 = 0.517993283065282,$
$C_4 = 0.130265924169716, C_5 = 0.0735482263924198.$

So, we obtain the following approximate solution of the FIE (8),

$$S_5(x) = 1.00029591104090 + 0.996100458779729x + 0.517993283065282x^2 + 0.130265924169716\,x^3 + 0.0735482263924198x^4.$$

Table 4 shows a comparison of the exact (analytical) solution $\varphi(x)$ of the FIE (8) against the numerical solution $S_n(x)$ for different values of x taken in the given interval [0, 1]. The error computed as the absolute value of the difference between the exact solution and numerical solution as shown in the fourth column of Table 4.

Table 4: The exact and numerical solutions of the FIE (8) using collocation method for n = 5.

x	$\varphi(x)$ Analytical solution	$S_5(x)$ Numerical solution	$\lvert\varphi(x)-S_5(x)\rvert$ Error
0	1	1.00029591104090	0.000295911040899099
0.25	1.284025417	1.28401830875191	0.00000710824809146970
0.5	1.648721271	1.64872446586782	0.00000319486782474065
0.75	2.117000017	2.11696953211599	0.0000304848840078087
1	2.718281828	2.71820380344805	0.0000780245519536038

Example 4: Solve the following integral equation using the collocation method

$$\varphi(x)-\int_0^1 x^3 e^{xy}\varphi(y)dy = x^2 - x^2e^x + 2x\,e^x - 2e^x + 2\ , \qquad (9)$$

where the exact solution is given as φ $(x) = x^2$.

Solution: By following the same steps in the last two examples, we solve this integral equation. Thus we obtain the results shown in Table 5. This table shows a comparison of the exact (analytical) solution $\varphi(x)$ of the FIE (9) against the numerical solution $S_n(x)$ for different values of x taken in the given interval [0, 1]. The error can be determined as the absolute value of the difference between the exact solution and numerical solution as shown in the fourth column of Table 5.

Table 5: The exact and numerical solutions of the FIE (9) using collocation method.

x	$\varphi(x)$ Analytical solution	$S_5(x)$ Numerical solution	$\lvert\varphi(x)-S_5(x)\rvert$ Error
0	0	0	0
1/7	0.02040816327	10.020408163265306	$3.46944695195361\times10^{-18}$
2/7	0.08163265306	50.081632653061224	$1.38777878078145\times10^{-17}$
3/7	0.1836734694	0.183673469387755	$2.77555756156289\times10^{-17}$
4/7	0.3265306122	0.326530612244898	0
5/7	0.5102040816	0.510204081632653	0
6/7	0.7346938776	10.73469387755102	$1.11022302462516\times10^{-16}$

.

4. The Galerkin method

Here we show how to apply the Galerkin method [18, 19] to solve FIE (2), to do so we approximate the exact solution $\varphi(x)$ in the form

$$S_n(x)=\sum_{i=1}^{n} C_i\,\psi_i(x)\ . \tag{10}$$

Such that the functions $\psi_1(x),\psi_2(x),\ .\ .\ .\ ,\psi_n(x)$ are linearly independent on the interval (a,b). Substituting the approximate solution (10) in the FIE (2) leads to

$$S_n(x)=f(x)+\int_a^b K(x,y)\,S_n(y)\,dy+E\left(x,C_1,C_2,...,C_n\right). \tag{12}$$

Conceptually, the Galerkin method depends on n conditions to find the constants $C_1,C_2,\ldots,C_n$ of approximate solution $S_n(x)$. For this, we shall emphasize that the functions $\psi_1(x),\psi_2(x),\ .\ .\ .\ ,\psi_n(x)$ need to satisfy the following orthogonal relation,

$$\int_a^b \psi_i(x)\,E\left(x,C_1,C_2,...,C_n\right)dx=0\quad ,i=1,2,...,n\ . \tag{12}$$

The formula (12) is called the basic rule of Galerkin method. Substituting equation (10) in the formula (12) leads to the following n algebraic linear equations in n unknowns $(C_1,C_2,\ldots,C_n)$ as,

$$\int_a^b \psi_i(x)\left[S_n(x)-f(x)-\int_a^b K(x,y)\,S_n(y)dy\right]dx=0\ . \tag{13}$$

A final step is to solve the resulting linear system (13) and obtain the approximate solution $S_n(x)$. Next we give some demonstration examples showing the method.

Example 5: Resolve the integral equation (6) using the Galerkin method.

Solution: Suppose that the approximate solution takes the form

$$S_n(x)=\sum_{k=1}^{n} C_k\ \psi_k(x),$$

where we set

$$n=5\,,\quad \psi_1(x)=1,\quad \psi_2(x)=x\ ,\quad \psi_3(x)=x^2,\quad \psi_4(x)=x^3,\quad \psi_5(x)=x^4.$$

Substituting this approximate solution in the FIE (6) leads to,

$$S_n(x)+2\int_0^1 e^{x-y}S_n(y)dy=2x\ e^x+E(x\ ,C_1\ ,C_2\ ,C_3,C_4\ ,C_5)\ .$$

That is

$$\left(C_1+C_2x+C_3x^2+C_4x^3+C_5x^4\right)+2\int_0^1 e^{x-y}\left(C_1+C_2y+C_3y^2+C_4y^3+C_5y^4\right)dy=$$
$$2xe^x+E(x,C_1,C_2,C_3,C_4,C_5).$$

Such that the last term represents the error. We have the following orthogonal relation

$$\int_0^1 \psi_i(x)\left(S_5(x)-\int_0^1 e^{x-y}S_5(y)\,dy-2xe^x\right)dx=0.$$

Thus we obtain the following linear system

$$3.172322538C_1+1.408081421C_2+0.885255052C_3+0.64152406C_4+0.50185506C_5=2.$$

$$1.764241118C_1+0.861815568C_2+0.571205588C_3+0.42785788C_4+0.34233931C_5=1.436563656\ .$$
$$10.241414754C_1+0.629599186C_2+0.430716134C_3+0.33033285C_4+0.26903951C_5=1.12687269.$$

$$0.962319396C_1+0.497766101C_2+0.34764558C_3+0.27124052C_4+0.2239804C_5=0.92907290\ .$$

$$0.78728608C_1+0.41216593C_2+0.29206880C_3+0.2308482C_4+0.192718C_5=0.7911991\ .$$

By using Maple18 we solve this system, and we get

$$C_1=-0.652941489154673,\ C_2=1.05430732275121, C_3=2.95705864541767,$$
$$C_4=-1.22205159243499, C_4=1.50046278908200.$$

So, we obtain the following approximate solution of the FIE (6),

$$S_5(x)=-0.652941489154673+1.05430732275121x+2.95705864541767x^2$$
$$-1.22205159243499\,x^3+1.50046278908200x^4.$$

Table 6 shows a comparison of the exact (analytical) solution $\varphi(x)$ of the FIE (6) against the numerical solution $S_n(x)$ for different values of x taken in the given interval [0, 1]. The error can be determined as the absolute value of the difference between the exact solution and numerical solution as shown in the fourth column of Table 5. By increasing the number of pointsfrom n = 5 to n = 7, we obtain more accurate solution of the FIE (6) as shown in the fourth column of Table 7.

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Table 6: The exact and numerical solutions of the FIE (6) using Galerkin method for n=5.

x	$\varphi(x)$ Analytical solution	$S_5(x)$ Numerical solution	$\lvert\varphi(x)-S_5(x)\rvert$ Error
0	-0.6666666667	- 0.652941489154673	0.0137251775119934
0.25	-0.2140042362	0.217781866490212	0.00377763029021158
0.5	0.5495737569	0.554499308838600	0.00492555193860045
0.75	1.764166681	1.76033727975564	0.00382940124436160
1	3.624375770	3.63683567566122	0.0124599056612165

Table 7: The exact and numerical solutions of the FIE (6) using Galerkin method for n=7.

x	$\varphi(x)$ Analytical solution	$S_5(x)$ Numerical solution	$\lvert\varphi(x)-S_7(x)\rvert$ Error
0	-0.6666666667	0.66611997336096 -5	0.000546693305702117
0.25	-0.2140042362	0.21397167787551 -4	0.0000325583244857319
0.5	0.5495737569	0.54958130831820 6	0.00000755141820640848
0.75	1.764166681	1.76411535722256	0.0000513237774431019
1	3.624375770	3.62377018995952	0.000605580040478504

Example 6: Resolve the FIE (7) by using the Galerkin method.
Solution: Suppose that the approximate solution of the FIE (7) takes the form

$$S_n(x) = \sum_{k=1}^{n} C_k \ \psi_k(x),$$

where we set

$$n = 5, \quad \psi_1(x) = 1, \ \psi_2(x) = x \ , \ \psi_3(x) = x^2, \ \psi_4(x) = x^3, \ \psi_5(x) = x^4.$$

Hence, one has

$$S_5(x) - \int_0^{\frac{\pi}{2}} x^5 \sin(xy)\, S_5(y)\, dy = 3x^2 - 2x^2\cos\left(\frac{\pi}{2}x\right) + \frac{1}{4}\pi^2 x^4 \cos\left(\frac{\pi}{2}x\right) - \pi x^3 \sin\left(\frac{\pi}{2}x\right) + \mathrm{E}(x, C_1, \ldots, C_n).$$

We have the following orthogonal relation

$$\int_0^{\frac{\pi}{2}} \psi_i(x)\left(S_5(x)-\int_0^{\frac{\pi}{2}} x^5 \sin(xy)\, S_5(y)\, dy -3x^2+2x^2\cos\left(\frac{\pi}{2}x\right)-\frac{\pi^2}{4}x^4 \cos\left(\frac{\pi}{2}x\right)+\pi x^3 \sin\left(\frac{\pi}{2}x\right)\right) dx = 0.$$

Thus we obtain the following linear system,

For $i = 1$, one has

$1.19566\, C_1 + 1.39295 C_2 + 1.59690 C_3 + 1.92839 C_4 + 2.43118 C_5 = 1.59690.$

For $i = 2$, one has

$2.5200\, C_1 + 2.25855 C_2 + 2.37165 C_3 + 2.72760 C_4 + 3.32773 C_5 = 2.37165.$

For $i = 3$, one has

$3.89440\, C_1 + 3.36792 C_2 + 3.43716 C_3 + 3.86006 C_4 + 4.61448 C_5 = 3.43716 .$

For $i = 4$, one has

$5.74584\, C_1 + 4.92165 C_2 + 4.95825 C_3 + 5.49120 C_4 + 6.47331 C_5 = 4.95825.$

For $i = 5$, one has

$8.38895\, C_1 + 7.16139 C_2 + 7.16346 C_3 + 7.86107 C_4 + 9.17233 C_5 = 7.16346.$

we solve this linear system for the unknowns $C_1,\ C_2,\ C_3,\ C_4$, C_5 to obtain,8By using Maple1

$C_1 = 0.0000362,\ C_2 = -0.0004355, C_3 = 1.0012002,\ C_4 = -0.0011562,$

$C_5 = 0.0003604,$

So, we obtain the following approximate solution,

$S_5(x) = 0.0000362 - 0.0004355x + 1.0012002x^2 - 0.0011562x^3 + 0.0003604x^4$

Table 8: The exact and numerical solutions of the FIE (7) using Galerkin method.

x	$\varphi(x)$ Analytical solution	$S_5(x)$ Numerical solution	$\lvert\varphi(x)-S_5(x)\rvert$ Error
0	0	0.000036264	3.6264199×10^{-5}
$\pi/14$	0.050355124	0.050341947	1.3176878×10^{-5}
$\pi/7$	0.201420498	0.201413158	7.3399980×10^{-6}
$3\pi/14$	0.453196120	0.453204407	8.2871713×10^{-6}
$2\pi/7$	0.805681992	0.805692146	1.0153799×10^{-5}
$5\pi/14$	1.258878113	1.258874759	3.353460×10^{-6}
$3\pi/7$	1.812784482	1.812772572	1.191060×10^{-5}
$\pi/2$	2.467401101	2.467427845	2.674482×10^{-5}

Table 8 shows the values of the exact (analytical) solution $\varphi(x)$ of the FIE (7) against the numerical solution $S_n(x)$ for different values of x taken in the given interval $[0, \pi/2]$. The error can be

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determined as the absolute value of the difference between the exact solution and the numerical solution as shown in the fourth column of Table 8.

Example 7: Resolve the FIE (8) by using the Galerkin method.

Solution: Suppose that the approximate solution of the FIE (8) takes the form

$$S_n(x)=\sum_{k=1}^{n} C_k \ \psi_k(x),$$

where we set

$$n=5,\ \ \psi_1(x)=1,\ \psi_2(x)=x\ ,\ \psi_3(x)=x^2,\ \ \psi_4(x)=x^3,\ \ \psi_5(x)=x^4.$$

Hence, one has

$$S_5(x)-\int_0^1 (x+1)^3 e^{xy} S_5(y)dy \ =e^x \text{-}(x+1)^2\left(e^{x+1}-1\right)+\mathrm{E}\left(x,C_1,\ldots,C_5\right).$$

We have the following orthogonal relation

$$\int_0^1 \psi_i(x)\left(S_5(x)-\int_0^1 (x+1)^3 e^{xy} S_5(y)dy - e^x+(x+1)^2\left(e^{x+1}-1\right)\right)dx \ \ =0.$$

we solve the resulting linear system for the unknowns $C_1,\ C_2,\ C_3,\ C_4,\ C_5$ to 8By using Maple1 obtain,

$C_1=0.999578591660359\ ,C_2=1.00770410360354\ ,C_3=0.469589761215411$,
$C_4=0.202697460126136$, $C_5=0.0382344696324625$,

So, we obtain the following approximate solution of the FIE (8),

$$S_5(x)=\ 0.999578591660359\ -1.00770410360354x\ +\ 0.469589761215411x^2+$$
$$0.202697460126136\,x^3+0.0382344696324625x^4$$

Table 9: The exact and numerical solutions of the FIE (8) using collocation method for n = 5.

x	$\varphi(x)$ Analytical solution	$S_5(x)$ Numerical solution	$\lvert\varphi(x)-S_5(x)\rvert$ Error
0	1	0.999578591660359	0.000421408339641327
0.25	1.284025417	1.28417047884868	0.000145061848680461
0.5	1.648721271	1.64855492063378	0.000166350366220591
0.75	2.117000017	2.11711152619455	0.000111509194545167
1	2.718281828	2.71780438623791	0.000477441762088304

Table 9 shows the values of the exact (analytical) solution $\varphi(x)$ of the FIE (8) against the numerical solution $S_n(x)$ for different values of x taken in the given interval [0, π/2]. The error can be determined as the absolute value of the difference between the exact solution and the numerical solution as shown in the fourth column of Table 9.

Example 8: Resolve the integral equation (9) using the Galerkin method.

Solution: By following the same steps in the last two examples, we solve this integral equation. Thus we obtain the results shown in Table 10. This table shows the values of the exact (analytical) solution $\varphi(x)$ of the FIE (9) against the numerical solution $S_n(x)$ for different values of x taken in the given interval [0, 1] for n = 5. The error can be determined as the absolute value of the difference between the exact solution and numerical solution as shown in the fourth column of Table 10.

Table 10: The exact and numerical solutions of the FIE (9) using the Galerkin method.

x	$\varphi(x)$ Analytical solution	$S_5(x)$ Numerical solution	$\lvert\varphi(x)-S_5(x)\rvert$ Error
0	0	0	0
1/7	0.02040816327	0.0204081632653062	$5.89805981832114\times10^{-17}$
2/7	0.08163265306	0.0816326530612247	$1.94289029309402\times10^{-16}$
3/7	.18367346940	0.183673469387755	$3.05311331771918\times10^{-16}$
4/7	0.3265306122	0.326530612244898	$2.77555756156289\times10^{-16}$
5/7	0.5102040816	0.510204081632653	$1.11022302462516\times10^{-16}$
6/7	0.7346938776	0.734693877551020	$1.11022302462516\times10^{-16}$

5. Discussion and conclusion

In this paper we implemented both the collocation and the Galerkin methods to numerically solve the second kind FIEs (6), (7), (8) and (9) of continuous kernel. We measure the error of the computations as the absolute value of the difference between the exact (analytical) solution $\varphi(x)$ and numerical solution $S_5(x)$. Hence, comparing the error of the numerical computations using the collocation method shown in the fourth column of Tables 1, 2, 3, 4, 5 against the corresponding

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computations using the Galerkin method shown in the fourth column of Tables 5, 6, 7, 8, 9, 10 one should notice that the computation error of the collocation method is much smaller than the one for the Galerkin method. Therefore at least for the second kind FIEs (6), (7), (8) and (9) that we consider here and for considerably a few interpolation points ($n=5$) in the given interval, one could claim that the collocation method performs better than the Galerkin method. Therefore one can conclude that the illustrated example for second kind Fredholm integral equation of continous kernel shows that the collocation method seems to converge faster than the Galerkin method.

Conceptually, the difference in the performance between the collocation and Galerkin method is due to the fact that, in the collocation method we force the error to vanish at the collocated points $x_1, x_2, \ldots, x_n$. Whereas in the Galerkin method we make the error orthogonal to n given linearly independent functions $\psi_1(x), \psi_2(x), \ldots, \psi_n(x)$ on an interval (a,b).

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