Certain Subclass of Meromorphic Valent Functions Defined by Convolution with Positive Coefficients

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Abstract

In this paper we introduce and study a subclass $R_p(f,g,\alpha,\zeta,\beta)$ of meromorphic $p-$ valent functions in $\mathbb{U}^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$. Therefore we obtain some coefficient estimates, distortion theorems, convex linear combinations and radius of convexity for functions belonging to the subclass $R_p(f,g,\alpha,\zeta,\beta)$. Also we derive several interesting results involving Hadamard product (or convolution) of functions belonging to this subclass.

Keywords: *$p-$ valent Meromorphic functions, Convolution.*

1. Introduction

The class of meromorphic functions which are analytic and p -valent in $\mathbb{U}^* = \{z \in \mathbb{C}: 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$ and has the form:

$$f(z) = z^{-p} + \sum_{n=0}^{\infty} a_{p+n} z^{p+n}, \quad \left(a_{p+n} \geq 0, p \in \mathbb{N} = \{1,2,\dots\}\right), \qquad (1.1)$$

is denoted by $\sum_p^*$. For $f(z)$ in this form and $g(z) \in \sum_p^*$ given by

$$g(z) = z^{-p} + \sum_{n=0}^{\infty} b_{p+n} z^{p+n}, \quad (p \in \mathbb{N}), \qquad (1.2)$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=z^{-p}+\sum_{n=0}^{\infty}a_{p+n}b_{p+n}z^{p+n}=\ (g*f)(z). \qquad (1.3)$$

For $0\leq\alpha<p, 0<\beta\leq 1, \frac{1}{2}\leq\zeta\leq 1, p\in\mathbb{N}$ and $g(z)\in\sum_p^*$ we say that $f(z)\in\ R_p(f,g,\alpha,\zeta,\beta)$ if it satisfies:

$$\left|\frac{z^{p+1}(f*g)'(z)+p}{(2\zeta-1)z^{p+1}(f*g)'(z)+(2\zeta\alpha-p)}\right|<\beta(z\in\mathbb{U}^{*}). \qquad (1.4)$$

$$g(z)=z^{-p}+\sum_{n=0}^{\infty}\left(\frac{p+\lambda n}{p}\right)^{k}z^{p+n}, (\lambda\geq 0, p\in\mathbb{N}, k\in\mathbb{N}_0=\mathbb{N}\backslash\{0\}).$$

Putting $g(z)=z^{-p}+\sum_{n=0}^{\infty}\left(\frac{p+\lambda n}{p}\right)^{k}z^{p+n}, (\lambda\geq 0, p\in\mathbb{N}, k\in\mathbb{N}_0=\mathbb{N}\backslash\{0\})$ in (1.4), we obtain the class $F_{p,\lambda}^{m}(\alpha,\beta,\zeta)$ studied earlier (see [2]).

We also note that, for different choices of $g(z)$ we have the following new classes:

(i) $R_p\left(\frac{z^{-p}}{1-z},\alpha,\zeta,\beta\right)=R_{p,\zeta}(,\alpha,\beta)$

$$\left\{f\in\sum_p^*:\begin{array}{c}\left|\frac{z^{p+1}\left(f(z)\right)'+p}{(2\zeta-1)z^{p+1}\left(f(z)\right)'+(2\zeta\gamma-p)}\right|<\beta\\(0\leq\gamma<p;\ 0<\beta\leq 1;\frac{1}{2}\leq\zeta\leq 1)\end{array}\right\};$$

(ii) $R_p(z^{-p}+\sum_{n=1}^{\infty}\Gamma_n(\alpha_1)z^n;\alpha,\zeta,\beta)=R_{p,q,s}(\alpha_1,\gamma,\zeta,\beta)$

$$\left\{f\in\sum_p^*:\begin{array}{c}\left|\frac{z^{p+1}\left(H_{p,q,s}(\alpha_1)f(z)\right)'+p}{(2\zeta-1)z^{p+1}\left(H_{p,q,s}(\alpha_1)f(z)\right)'+(2\zeta\gamma-p)}\right|<\beta\\(\alpha_j>0(j=1,2,\ldots,q),\beta_j>0(j=1,2,\ldots,s)\\0\leq\gamma<p;\ 0<\beta\leq 1;\frac{1}{2}\leq\zeta\leq 1)\end{array}\right\};$$

Where

$$\Gamma_m(\alpha_1) = \frac{(\alpha_1)_m \dots (\alpha_q)_m}{(\beta_1)_m \dots (\beta_s)_m} \frac{1}{m!}$$

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & k = 0 \\ a(a+1)(a+2)\dots(a+k-1) & k \in \mathbb{N} \end{cases},$$

and $H_{p,q,s}(\alpha_1)$ was introduced and studied by Liu-Srivastava [8];

(iii) $R_p\left(z^{-p} + \sum_{n=1}^{\infty} \frac{(\beta_1)_{n+p} \dots (\beta_s)_{n+p} (\lambda+p)_{n+p}}{(\alpha_1)_{n+p} \dots (\alpha_l)_{n+p}} z^n; \alpha, \zeta, \beta\right) = R_{p,\lambda}(\alpha_1, \gamma, \zeta, \beta)$

$$\left\{ f \in \sum\nolimits_p^* : \begin{array}{c} \left| \dfrac{z^{p+1}\left(M_{p,l,s}^{\lambda}(\alpha_1) f(z)\right)' + p}{(2\zeta-1) z^{p+1}\left(M_{p,l,s}^{\lambda}(\alpha_1) f(z)\right)' + (2\zeta\gamma - p)} \right| < \beta \\ (\beta_j > 0 (j = 1,2,\dots,s), \alpha_j > 0 (j = 1,2,\dots,l), \\ 0 \le \gamma < p;\ 0 < \beta \le 1; \frac{1}{2} \le \zeta \le 1; \lambda \ge p; z \in \mathbb{U}^*) \end{array} \right\},$$

where the operator $M_{p,l,s}^{\lambda}(\alpha_1)$ was introduced and studied by Mostafa [9, with $m = 0$];

(iv) $R_p\left(z^{-p} + \frac{\Gamma(\nu)}{\Gamma(\lambda+\upsilon)} \sum_{n=1}^{\infty} \frac{\Gamma(\lambda+\upsilon+n)(\mu)_n}{\Gamma(k+\upsilon)(1)_n} z^{n-p}; \alpha, \zeta, \beta\right) = R_{p,\mu}^{\lambda,\nu}(\gamma, \zeta, \beta)$

$$\left\{ f \in \sum\nolimits_p^* : \begin{array}{c} \left| \dfrac{z^{p+1}\left(H_{p,\upsilon,\mu}^{\lambda} f(z)\right)' + p}{(2\zeta-1) z^{p+1}\left(H_{p,\upsilon,\mu}^{\lambda} f(z)\right)' + (2\zeta\gamma - p)} \right| < \beta \\ (\lambda \ge 0;\ \mu > 0; \nu > -1;\ 0 \le \gamma < p;\ 0 < \beta \le 1; \\ \frac{1}{2} \le \zeta \le 1; z \in \mathbb{U}^*) \end{array} \right\},$$

where the operator $H_{p,\upsilon,\mu}^{\lambda}$ was introduced and studied by Mostafa [10].

For more details of meromorphic multivalent functions (see [1], [3], [4], [5], [6], [7], [11] and [13]).

2. Coefficient Estimates

In the reminder of this paper we assume that:

$$0 \le \alpha < p, 0 < \beta \le 1, \frac{1}{2} \le \zeta \le 1, b_{p+n} > 0, n \in \mathbb{N}_0 \text{ and } p \in \mathbb{N}.$$

Theorem1 . *A function* $f(z) \in R_p(g,\alpha,\zeta,\beta)$ *if and only if*

$$\sum_{n=0}^{\infty}(p+n)(1+2\beta\zeta-\beta)a_{p+n}b_{p+n} \le 2\beta\zeta(p-\alpha) \qquad (2.1)$$

Proof: Suppose (2.1) holds. For $|z| = r < 1$, we have

$$|z^{p+1}(f*g)'(z)+p| - \beta|(2\zeta-1)z^{p+1}(f*g)'(z)+(2\zeta\alpha-p)|$$

$$= \left|\sum_{n=0}^{\infty}(p+n)a_{p+n}b_{p+n}z^{2p+n}\right| - \beta\left|2\zeta(p-\alpha) - \sum_{n=0}^{\infty}(p+n)(2\zeta-1)a_{p+n}b_{p+n}z^{2p+n}\right|$$

$$\le \sum_{n=0}^{\infty}(p+n)a_{p+n}b_{p+n}r^{2p+n} - 2\beta\zeta(p-\alpha) + \sum_{n=0}^{\infty}\beta(p+n)(2\zeta-1)a_{p+n}b_{p+n}r^{2p+n}$$

$$< \sum_{n=0}^{\infty}(p+n)(1+2\beta\zeta-\beta)a_{p+n}b_{p+n}r^{2p+n} - 2\beta\zeta(p-\alpha) \le 0.$$

Hence $f(z) \in R_p(g,\alpha,\zeta,\beta)$.

Conversely, suppose that

$$\left|\frac{(f*g)'(z)+p}{(2\zeta-1)z^{p+1}(f*g)'(z)+(2\zeta\alpha-p)}\right|$$

$$\left|\frac{\sum_{n=0}^{\infty}(p+n)a_{p+n}b_{p+n}z^{2p+n}}{(2\zeta-1)(p-\sum_{n=0}^{\infty}(p+n)a_{p+n}b_{p+n}z^{2p+n})-(2\zeta\alpha-1)}\right| < \beta\left(z \in \mathbb{U}^*\right).$$

Using the fact that $Re(z) \le |z|$ for all z, we have

$$Re\left\{\frac{\sum_{n=0}^{\infty}(p+n)a_{p+n}b_{p+n}z^{2p+n}}{2\zeta(p-\alpha)-\sum_{n=0}^{\infty}(p+n)(2\zeta-1)a_{p+n}b_{p+n}z^{2p+n}}\right\}<\beta\left(z\in\mathbb{U}^{*}\right). \quad (2.2)$$

Now choose the values of z on the real axis so that $z^{p+1}(f*g)'(z)$ is real. Upon clearing the denominator (2.2) and $z\to 1^{-}$ through positive values, we obtain (2.1).

Corollary1 . *If* $f(z)\in R_p(g,\alpha,\zeta,\beta)$. *Then*

$$a_{p+n}\leq\frac{2\beta\zeta(p-\alpha)}{(p+n)(1+2\beta\zeta-\beta)b_{p+n}} \quad (2.3)$$

Sharpness holds for

$$f(z)=z^{-p}+\frac{2\beta\zeta(p-\alpha)}{(p+n)(1+2\beta\zeta-\beta)b_{p+n}} \quad (2.4)$$

3. Distortion Theorems

Theorem 2 . *If* $f(z)\in R_p(g,\alpha,\zeta,\beta)$. *Then*

$$\left(\frac{(p+m-1)!}{(p-1)!}-\frac{(p-1)!}{(p-m)!}\cdot\frac{2\beta\zeta(p-\alpha)}{(1+2\beta\zeta-\beta)b_p}r^{2p}\right)r^{-(p+m)}$$

$$\leq\left|f^{(m)}(z)\right|\leq\left(\frac{(p+m-1)!}{(p-1)!}+\frac{(p-1)!}{(p-m)!}\cdot\frac{2\beta\zeta(p-\alpha)}{(1+2\beta\zeta-\beta)b_p}r^{2p}\right)r^{-(p+m)}$$

$$\left(0<|z|=r<1; m\in\mathbb{N}_0; p\in\mathbb{N}, b_{n+p}>0; p>m\right).$$

Sharpness holds for

$$f(z)=z^{-p}+\frac{2\beta\zeta(p-\alpha)}{p(1+2\beta\zeta-\beta)b_p}z^{p}.$$

Proof. In view of (2.1), we have

$$\frac{(1+2\beta\zeta-\beta)b_p}{p!}\sum_{n=0}^{\infty}(p+n)!\,a_{p+n} \le \sum_{n=0}^{\infty}(p+n)(1+2\beta\zeta-\beta)a_{p+n}b_{p+n} \quad \le 2\beta\zeta(p-\alpha),$$

which yields

$$\sum_{n=0}^{\infty}(p+n)!\,a_{p+n} \le \frac{2\beta\zeta(p-\alpha)p!}{(1+2\beta\zeta-\beta)b_p}. \qquad (3.1)$$

Now, differentiating both sides of (1.1) m -times with respect to z , we have

$$f^m(z) = (-1)^m\frac{(p+m-1)!}{(p-1)!}z^{-(p+m)} + \sum_{n=0}^{\infty}\frac{(p+n)!}{(p+n-m)!}a_{p+n}z^{p+n-m} \ (m \in \mathbb{N}_0; p \in \mathbb{N}; p+n > m) \qquad (3.2)$$

from (3.1) and (3.2), we have

$$|f^m(z)| \le \frac{(p+m-1)!}{(p-1)!}r^{-(p+m)} + r^{p-m}\sum_{n=0}^{\infty}\frac{(p+n)!}{(p+n-m)!}a_{p+n}$$

$$\le \left(\frac{(p+m-1)!}{(p-1)!} + \frac{p!}{(p-m)!}.\frac{2\beta\zeta(p-\alpha)}{(1+2\beta\zeta-\beta)b_p}r^{2p}\right)r^{-(p+m)}$$

and

$$|f^m(z)| \ge \frac{(p+m-1)!}{(p-1)!}r^{-(p+m)} - r^{p-m}\sum_{n=0}^{\infty}\frac{(p+n)!}{(p+n-m)!}a_{p+n}$$

$$\ge \left(\frac{(p+m-1)!}{(p-1)!} - \frac{p!}{(p-m)!}.\frac{2\beta\zeta(p-\alpha)}{(1+2\beta\zeta-\beta)b_p}r^{2p}\right)r^{-(p+m)}.$$

Theorem 2 is completed.

4. Radius of Convexity

Theorem 3. Let $f(z) \in R_p(g,\alpha,\zeta,\beta)$. Then $f(z)$ is meromorphically $p-$ valent convex of order σ $(0 \leq \sigma < p)$ in $|z| < r$ where

$$r = \inf_{n} \left\{ \frac{(p-\sigma)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\alpha)(n+3p-\sigma)} \right\}^{\frac{1}{2p+n}} (n \geq 0; p \in \mathbb{N}). \quad (4.1)$$

The function $f(z)$ given by (2.4) gives the sharpness.

Proof. We must show that

$$\left| \frac{(zf'(z))' + pf'(z)}{f'(z)} \right| \leq (p-\sigma) \text{ for } |z| < r,$$

where r is given by (4.1). Indeed we find from (1.1) that

$$\left| \frac{(zf'(z))' + pf'(z)}{f'(z)} \right| \leq \frac{\sum_{n=0}^{\infty}(p+n)(2p+n)a_{p+n}|z|^{2p+n}}{p - \sum_{n=0}^{\infty}(p+n)a_{p+n}|z|^{2p+n}}.$$

Thus

$$\left| \frac{(zf'(z))' + pf'(z)}{f'(z)} \right| \leq p - \sigma,$$

if

$$\sum_{n=0}^{\infty} \left(\frac{(p+n)(n+3p-\sigma)}{(p-\sigma)} \right) a_{p+n}|z|^{2p+n} \leq 1. \quad (4.2)$$

But, by Theorem1 , (4.2) will be true if

$$\left(\frac{(n+3p-\sigma)}{(p-\sigma)} \right) |z|^{2p+n} \leq \frac{(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\alpha)},$$

that is, if

$$|z| \leq \left\{\frac{(p-\sigma)(1+2\beta\zeta-\beta)b_{p+n}}{(p+n)(n+3p-\sigma)}\right\}^{\frac{1}{2p+n}} (n \geq 0; p \in \mathbb{N}). \quad (4.3)$$

Theorem 3 follows easily from (4.3).

5. Closure Theorems

Theorem 4. Let

$$f_{p-1}(z) = z^{-p} \quad (5.1)$$

and

$$f_{p+n}(z) = z^{-p} + \frac{2\beta\zeta(p-\alpha)}{(p+n)(1+2\beta\zeta-\beta)b_{p+n}} z^{p+n}(n \geq 0;\ p \in \mathbb{N} \quad (5.2)$$

Then $f(z) \in R_p(g,\alpha,\zeta,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=-1}^{\infty} \psi_{p+n} f_{p+n}(z), \quad (5.3)$$

where $\psi_{p+n} \geq 0$ and $\sum_{n=-1}^{\infty} \psi_{p+n} = 1$.

Proof. Suppose that $f(z) = \sum_{n=-1}^{\infty} \psi_{p+n} f_{p+n}(z)$. Then

$$f(z) = z^{-p} + \sum_{n=0}^{\infty} \frac{2\beta\zeta(p-\alpha)}{(p+n)(1+2\beta\zeta-\beta)b_{p+n}} \psi_{p+n} z^{p+n}.$$

We have

$$\sum_{n=0}^{\infty} (p+n)(1+2\beta\zeta-\beta)b_{p+n}. \frac{2\beta\zeta(p-\alpha)}{(p+n)(1+2\beta\zeta-\beta)b_{p+n}} \psi_{p+n}.$$

$$= 2\beta\zeta(p-\alpha) \sum_{n=0}^{\infty} \psi_{p+n} = 2\beta\zeta(p-\alpha)\left(1-\psi_{p+n}\right)$$

$$\leq 2\beta\zeta(p-\alpha).$$

From Theorem 1, $f(z) \in R_p(g,\alpha,\zeta,\beta)$.

Conversely, assume that $f(z)$ defined by (1.1) $\in R_p(g,\alpha,\zeta,\beta)$. Then (2.4) holds. Setting

$$\psi_{p+n} = \frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\alpha)} a_{p+n} (n \geq 0;\ p \in \mathbb{N}) \tag{5.4}$$

and

$$\psi_{p-1} = 1 - \sum_{n=0}^{\infty} \psi_{p+n},$$

we thus have (5.3). This evidently completes the proof of Theorem 4 .

Theorem 5. The class $R_p(g,\alpha,\zeta,\beta)$ is closed under convex linear combination.

Proof. Suppose each of the functions

$$f_i(z) = z^{-p} + \sum_{n=0}^{\infty} a_{p+n,i} z^{p+n}.\ \left(a_{p+n,i} \geq 0, i = 1,2\right), \tag{5.5}$$

are in the class $R_p(g,\alpha,\zeta,\beta)$. It is sufficient to show that the function $w(z)$ defined by

$$w(z) = (1-s)f_1(z) + sf_2(z)\ (0 \leq s \leq 1) \tag{5.6}$$

is also in the class $R_p(g,\alpha,\zeta,\beta)$. Since

$$w(z) = z^{-p} + \sum_{n=0}^{\infty} \left[(1-s)a_{p+n,1} + sa_{p+n,2}\right] z^{p+n} \quad (0 \leq s \leq 1).$$

In view of Theorem1 , we have

$$\sum_{n=1}^{\infty} (p+n)(1+2\beta\zeta-\beta)b_{p+n}\left[(1-s)a_{p+n,1} + sa_{p+n,2}\right]$$

$$= (1-s)\sum_{n=0}^{\infty}\left[(p+n)(1+2\beta\zeta-\beta)b_{p+n}\right]a_{p+n,1} + s\sum_{n=0}^{\infty}\left[(p+n)(1+2\beta\zeta-\beta)b_{p+n}\right]a_{p+n,2}$$

$$\leq (1-s)2\beta\zeta(p-\alpha) + 2s\beta\zeta(p-\alpha) = 2\beta\zeta(p-\alpha).$$

This shows that $w(z) \in R_p(g,\alpha,\zeta,\beta)$. and hence the proof of Theorem 5 is completed.

6. Convolution Properties

Theorem 6. Let $f_i(z) \in R_p(g,\alpha,\zeta,\beta)(i=1,2)$, where $f_i(z)(i=1,2)$ are in the form *(5.5)* . Then $(f_1 * f_2)(z) \in R_p(g,\omega,\zeta,\beta)$, where

$$\omega = p - \frac{2\beta\zeta(p-\alpha)^2}{p(1+2\beta\zeta-\beta)b_p}. \qquad (6.1)$$

Sharpness holds for

$$f_i(z) = z^{-p} + \frac{2\beta\zeta(p-\alpha)}{p(1+2\beta\zeta-\beta)b_p} z^p (i=1,2). \qquad (6.2)$$

Proof. Using the technique for analytic functions (see [12]), we need to find the largest real parameter ω such that

$$\sum_{n=0}^{\infty} \frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\omega)} a_{p+n,1}a_{p+n,2} \leq 1.$$

Since $f_i(z) \in R_p(g,\alpha,\zeta,\beta)(i=1,2)$, we have

$$\sum_{n=0}^{\infty} \frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\alpha)} a_{p+n,1} \leq 1$$

and

$$\sum_{n=0}^{\infty} \frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\alpha)} a_{p+n,2} \leq 1.$$

By Cauchy-Schwarz inequality we have

$$\sum_{n=0}^{\infty} \frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\alpha)} \sqrt{a_{p+n,1}a_{p+n,2}} \leq 1. \qquad (6.3)$$

Thus it is sufficient to show that

$$\frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\omega)}a_{p+n,1}a_{p+n,2}$$
$$\leq \frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\alpha)}\sqrt{a_{p+n,1}a_{p+n,2}}$$

or, equivalently, that

$$\sqrt{a_{p+n,1}a_{p+n,2}} \leq \frac{(p-\omega)}{(p-\alpha)}.$$

Hence, in the light of the inequality (6.3), it is sufficient to prove that

$$\frac{2\beta\zeta(p-\alpha)}{(p+n)(1+2\beta\zeta-\beta)b_{p+n}} \leq \frac{(p-\omega)}{(p-\alpha)} \tag{6.4}$$

It follows from (6.4) that

$$\omega = p - \frac{2\beta\zeta(p-\alpha)^2}{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}.$$

Let

$$G(n) = p - \frac{2\beta\zeta(p-\alpha)^2}{(p+n)(1+2\beta\zeta-\beta)b_{p+n}},$$

then $G(n)$ is increasing function of $n(n \geq 0)$. Therefore, we conclude that ,

$$\omega \leq G(0) = p - \frac{2\beta\zeta(p-\alpha)^2}{p(1+2\beta\zeta-\beta)b_p}$$

and hence the proof of Theorem 6 is completed.

Theorem 7. Let $f_1(z) \in R_p(g,\alpha,\zeta,\beta)$ and $f_2(z) \in R_p(g,\delta,\zeta,\beta)$ where $f_i(z)(i=1,2)$ are in the form *(5.5)* . Then $(f_1 * f_2)(z) \in R_p(g,\varphi,\zeta,\beta)$, where

$$\varphi = p - \frac{2\beta\zeta(p-\alpha)(p-\delta)}{p(1+2\beta\zeta-\beta)b_p}.$$

Sharpness holds for

$$f_1(z) = z^{-p} + \frac{2\beta\zeta(p-\alpha)}{p(1+2\beta\zeta-\beta)b_p} z^p$$

and

$$f_2(z) = z^{-p} + \frac{2\beta\zeta(p-\delta)}{p(1+2\beta\zeta-\beta)b_p} z^p$$

Proof. We need to find the largest real parameter φ such that

$$\sum_{n=0}^{\infty} \frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\varphi)} a_{p+n,1}a_{p+n,2} \leq 1.$$

Since $f_1(z) \in R_p(g,\alpha,\zeta,\beta)$, we have

$$\sum_{n=0}^{\infty} \frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\alpha)} a_{p+n,1} \leq 1,$$

and $f_2(z) \in R_p(g,\delta,\zeta,\beta)$, we have

$$\sum_{n=0}^{\infty} \frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\delta)} a_{p+n,2} \leq 1.$$

By Cauchy-Schwarz inequality we have

$$\sum_{n=0}^{\infty} \frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta\sqrt{(p-\alpha)}\sqrt{(p-\delta)}} \sqrt{a_{p+n,1}a_{p+n,2}} \leq 1. \qquad (6.5)$$

Thus it is sufficient to show that

$$\frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\varphi)} a_{p+n,1}a_{p+n,2}$$

$$\leq \frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta\sqrt{(p-\alpha)}\sqrt{(p-\delta)}} \sqrt{a_{p+n,1}a_{p+n,2}}$$

or, equivalently, that

$$\sqrt{a_{p+n,1}a_{p+n,2}} \leq \frac{(p-\varphi)}{\sqrt{(p-\alpha)}\sqrt{(p-\delta)}}.$$

Hence, in the light of the inequality (6.5), it is sufficient to prove that

$$\frac{2\beta\zeta\sqrt{(p-\alpha)}\sqrt{(p-\delta)}}{(p+n)(1+2\beta\zeta-\beta)b_{p+n}} \leq \frac{(p-\varphi)}{\sqrt{(p-\alpha)}\sqrt{(p-\delta)}}. \tag{6.6}$$

It follows from (6.6) that

$$\varphi \leq p - \frac{2\beta\zeta(p-\alpha)(p-\delta)}{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}.$$

Let

$$M(n) = p - \frac{2\beta\zeta(p-\alpha)(p-\delta)}{(p+n)(1+2\beta\zeta-\beta)b_{p+n}},$$

then $M(n)$ is increasing function of $n(n \geq 0)$. Therefore, we conclude that

$$\varphi \leq M(0) = p - \frac{2\beta\zeta(p-\alpha)(p-\delta)}{p(1+2\beta\zeta-\beta)b_p}$$

and hence the proof of Theorem 7 is completed.

Theorem 8. Let $f_i(z) \in R_p(g,\alpha,\zeta,\beta)(i=1,2)$ where $f_i(z)(i=1,2)$ are in the form *(5.5)* . Then

$$h(z) = z^{-p} + \sum_{n=1}^{\infty}\left({a^2}_{p+n,1} + {a^2}_{p+n,2}\right)z^{p+n},$$

belongs to the class $R_p(g,\alpha,\eta,\beta)$

$$\eta = p - \frac{4\beta\zeta(p-\alpha)^2}{p(1+2\beta\zeta-\beta)b_p}$$

Sharpness holds for $f_i(z)(i=1,2)$ defined by *(6.2).*

Proof. By using Theorem 1, we obtain

$$\sum_{n=0}^{\infty}\left\{\frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\alpha)}\right\}^2 a^2{}_{p+n,1}$$

$$\leq \sum_{n=0}^{\infty}\left\{\frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\alpha)}a_{p+n,1}\right\}^2 \leq 1 \qquad (6.7)$$

and

$$\sum_{n=0}^{\infty}\left\{\frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\alpha)}\right\}^2 a^2{}_{p+n,2}$$

$$\leq \sum_{n=0}^{\infty}\left\{\frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\alpha)}a_{p+n,2}\right\}^2 \leq 1 \qquad (6.8)$$

It follows from (6.7) and (6.8) that

$$\sum_{n=0}^{\infty}\frac{1}{2}\left\{\frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\alpha)}\right\}^2\left(a^2{}_{p+n,1}+a^2{}_{p+n,2}\right) \leq 1.$$

Therefore, we need to find the largest η such that

$$\frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\eta)} \leq \frac{1}{2}\left\{\frac{(p+n)(1+2\beta\zeta-\beta)b_{p+n}}{2\beta\zeta(p-\alpha)}\right\}^2.$$

Let

$$H(n) = p - \frac{2\beta\zeta(p-\alpha)^2}{(p+n)(1+2\beta\zeta-\beta)b_{p+n}},$$

then $H(n)$ is increasing function of $n(n \geq 0)$ Therefore, we conclude that

$$\eta \leq H(0) = p - \frac{2\beta\zeta(p-\alpha)^2}{p(1+2\beta\zeta-\beta)b_p}.$$

and hence the proof of Theorem 8 is completed.

Remarks (i) Putting $g(z) = z^{-p} + \sum_{n=0}^{\infty} \left(\frac{p+\lambda n}{p}\right)^{k} z^{p+n}$ $(\lambda \geq 0, p \in \mathbb{N}, k \in \mathbb{N}_0)$,

in our results we obtain the results obtained by Aouf [2].

(ii) Specializing the function $g(z)$ in our results, we obtain new resultes associated to the classes $R_{p,\zeta}(,\alpha,\beta)$, $R_{p,q,s}(\alpha_1,\gamma,\zeta,\beta)$ and $R_{p,\mu}^{\lambda,\nu}(\gamma,\zeta,\beta)$ defined in the introduction.

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