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Oscillation Theorems for Nonlinear Second Order Forced Differential Equations

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Abstract

Relating to the oscillation theory, in the present paper, we consider a class of forced nonlinear differential equations of second order. However, we discuss the problem of finding sufficient criteria for all solutions of these equations to be oscillate. By employing a generalized Ricati technique and also using an integral averaging technique, we derive several new oscillation theorems. Our results obtained here generalize and improve some of well-known ones in the literature. Some carefully selected examples are also given to illustrate the effect of impulses on the oscillatory behavior of all solutions for this class.

Keywords: *Oscillation; Forced Nonlinear differential equations of second order.*

1. Introduction

We consider the oscillation behavior of solutions of second order forced nonlinear differential equation

$$(r(t)\psi(x(t))\left|f(x'(t))\right|^{\alpha-1} f(x'(t)))' + q(t)g(x(t)) = H(t, x(t), x'(t)), \quad t \in [t_0, \infty), \qquad (1.1)$$

and

$$(r(t)\psi(x(t))f(x'(t)))' + q(t)g(x(t)) = H(t, x(t), x'(t)), \quad t \in [t_0, \infty), \qquad (1.2)$$

where $r, q \in C([t_0,\infty), \mathbb{R})$, and $f, \psi, g \in C(\mathbb{R}, \mathbb{R})$ and H is a continuous function on α is a positive real number. Throughout the paper, it is assumed that the following conditions are satisfied:

(A_1) $r(t) > 0, t \geq 0;$

(A$_2$) $xg(x) > 0,\ g \in C^1(\mathbb{R})\ \ for\ x \neq 0;$

(A$_3$) $\dfrac{H(t,x,y)}{g(x)} \leq p(t)\ \forall\ t \in [t_0, \infty)\ ;\ x, y \in \mathbb{R}$ and $x \neq 0.$

We restrict our attention only to the solutions of the differential equations (1.1) and (1.2) that exist on some ray $[t_0, \infty)$, where $t_0 \geq t$, to may depend on the particular solutions. Such a solution is said to be oscillatory if it has arbitrarily large zeros, and otherwise, it is said to be nonoscillatory. Equations (1.1) and (1.2) are called oscillatory if all its solutions are oscillatory.

The problem of finding oscillation criteria for second order nonlinear ordinary differential equations, which involve the average of integral of the alternating coefficient, has received the attention of many authors because in the fact that there are many physical systems are modeled by second order nonlinear ordinary differential equations; for example, the so called Emden – Fowler equation arises in the study of gas dynamics and fluid mechanics. This equation appears also in the study of relativistic mechanics, nuclear physics and in the study of chemically reacting systems.

The oscillatory theory as a part of the qualitative theory of differential equations has been developed rapidly in the last decades, and there has been a great deal of work on the oscillatory behavior of differential equations; see e.g. [1-35].

Remili [27], studied the equation

$$(r(t)x'(t))' + Q(t,x) = H(t, x'(t), x(t)), \tag{1.3}$$

and derived some oscillation criteria for the equation (1.3), where new results with additional suitable weighted function are investigated. Zhang and Wang [35], studied the following equation

$$(r(t)\psi(x(t))x'(t))' + Q(t,x) = H(t, x'(t), x(t)). \tag{1.4}$$

Temtek and Tiryaki [31] obtained several new oscillation results for the equation

$$(r(t)\psi(x(t))\left|x'(t)\right|^{\alpha-1} x'(t))' + Q(t,x) = H(t, x'(t), x(t)), \tag{1.5}$$

and its special cases by using generalized Riccati transformation and well known techniques.

In this paper, we continue in this direction the study of oscillatory properties of equations (1.1) and (1.2). The purpose of this paper is to improve and extend the above mentioned results. Our results are more general than the previous results. The relevance of our results becomes clear due to some carefully selected examples.

2. Main Results

In this section we prove our main results.

Theorem 2.1 Suppose that, conditions (A_1) - (A_3) hold, and

$$\frac{g'(x)}{\left(\psi(x)|g(x)|^{\alpha-1}\right)^{\frac{1}{\alpha}}} \geq k > 0 \text{ for all } x \in \mathbb{R} \ , \tag{2.1}$$

$$0 < k_1 \leq \frac{f(y)}{y} \leq k_2 \text{ for all } y = x'(t) \neq 0. \tag{2.2}$$

Let ρ be a positive continuously differentiable function over $[T,\infty)$ such that $\rho'(t) \geq 0$ over $[T_0,\infty)$;

$$\lim_{t\to\infty} \int_{T_0}^{t} \frac{1}{\left(\rho(s)r(s)\right)^{1/\alpha}} ds = \infty, \tag{2.3}$$

$$\limsup_{t\to\infty} \int_{T_0}^{t} Z(s)ds = \infty, \tag{2.4}$$

where $Z(s) = \rho(s)\left[(q(s)-p(s)) - \lambda r(s)\left(\frac{\rho'(s)}{\rho(s)}\right)^{\alpha+1}\right]$ and $\lambda = \frac{\alpha}{\alpha+1}(kk_1)^{\alpha+1}$,

Then all solutions of equation (1.1) are oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution on $[T,\infty)$, $T \geq T_0$ of the equation (1.1). We assume that $x(t)$ is positive on $[T,\infty)$, $T \geq t_0$. A similar argument holds for the case when $x(t)$ is negative. Let

$$w(t) = \frac{\rho(t)r(t)\psi(x(t))\left|f(x'(t))\right|^{\alpha-1} f(x'(t))}{g(x(t))} \ , \ t \geq T_0. \tag{2.5}$$

Then differentiating (2.5), (1.1) and take in account assumptions (A_1) - (A_3), (2.2) we have

$$w'(t) \le -\rho(t)[q(t)-p(t)] + \frac{\rho'(t)}{\rho(t)}|w(t)| - \frac{1}{k_2}\frac{\rho(t)r(t)\psi(x(t))\left|f(x'(t))\right|^{\alpha-1} f^2(x'(t))g'(x)}{g^2(x)}. \qquad (2.6)$$

In view of (2.1) we conclude that

$$w'(t) \le -\rho(t)[q(t)-p(t)] + \frac{\rho'(t)}{\rho(t)}|w(t)| - \frac{k}{k_2}\frac{|w(t)|^{\frac{\alpha+1}{\alpha}}}{\left(\rho(t)r(t)\right)^{1/\alpha}}. \qquad (2.7)$$

By using the extremum of one variable function it can be proved that

$$DX - EX^{\frac{\alpha}{\alpha+1}} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}} D^{\alpha+1}E^{-\alpha}, \;\; D \ge 0, E > 0, X \ge 0.$$

Now, by applying this inequality we have

$$\begin{aligned} w'(t) &\le -\rho(t)[q(t)-p(t)] + \lambda \frac{r(t)(\rho'(t))^{\alpha+1}}{(\rho(t))^\alpha} \\ &= -\rho(t)\left[q(t)-p(t)-\lambda r(t)\left(\frac{\rho'(t)}{\rho(t)}\right)^{\alpha+1}\right]. \end{aligned} \qquad (2.8)$$

Integrating (2.8) from T to t, we get

$$w(t) \le w(T) - \int_T^t \rho(s)\left[q(s)-p(s)-\lambda r(s)\left(\frac{\rho'(s)}{\rho(s)}\right)^{\alpha+1}\right] ds, \;\; t \ge T \ge T_0. \qquad (2.9)$$

Taking the limit for both sides of (2.9) and using (2.4), we find $w(t) \to -\infty$. Hence, there exists $T_1 \ge T$ such that $f(x'(t)) < 0 \Rightarrow x'(t) < 0, \;\; \forall t \ge T_1$.

Condition (2.4) also implies that $\int_T^\infty \rho(s)[q(s)-p(s)]ds = \infty$, and there exists $T_2 \ge T_1$ such that

$$\int_{T_1}^{T_2} \rho(s)[q(s)-p(s)]ds = 0 \text{ and } \int_{T_2}^{t} \rho(s)[q(s)-p(s)]ds \ge 0, \;\; \forall t \ge T_2 .$$

Multiplying Eq. (1.1) by $\rho(t)$ and integrating by parts on $[T_2, t]$, we have

$$\rho(t)[r(t)\psi(x(t))\left|f(x'(t))\right|^\alpha]' \le -\rho(t)g(x(t))[q(t)-p(t)].$$

Now, integrating by parts, we get

$$-\rho(t)\left[r(t)\psi(x(t))\left(-f(x'(t))\right)^{\alpha}\right]+C_{T_2}\le-\int_{T_2}^{t}\rho'(s)r(s)\psi(x(s))\left(-f(x'(s))\right)^{\alpha}ds$$
$$-\int_{T_2}^{t}\rho(s)g(x(s))[q(s)-p(s)]ds,$$

where

$$C_{T_2}=\frac{\rho(T_2)r(T_2)\psi(x(T_2))\left(-f(x'(T_2))\right)^{\alpha}}{g(x(T_2))}>0.$$

$$\rho(t)\left[r(t)\psi(x(t))\left(-f(x'(t))\right)^{\alpha}\right]\ge C_{T_2}+g(x(t))\int_{T_2}^{t}\rho(s)[q(s)-p(s)]ds$$
$$-\int_{T_2}^{t}x'(s)g'(x(s))\int_{T_2}^{s}\rho(u)[q(u)-p(u)]duds$$
$$+\int_{T_2}^{t}\rho'(s)r(s)\psi(x(s))\left(-f(x'(s))\right)^{\alpha}ds\le C_{T_2},\quad \forall\ t\ge T_1.$$

Therefore,

$$\rho(t)\left[r(t)\psi(x(t))\left(-f(x'(t))\right)^{\alpha}\right]\ge C_{T_2}$$

From (2.1) and (2.2), we find

$$\psi(x(t))\left(-f(x'(t))\right)^{\alpha}\ge\frac{C_{T_2}}{r(t)\rho(t)},$$

$$\left(\psi(x(t))\right)^{\frac{1}{\alpha}}\left(-f(x'(t))\right)\ge\left(\frac{C_{T_2}}{r(s)\rho(s)}\right)^{\frac{1}{\alpha}},$$

$$\int_{T_2}^{t}k_2\left(\psi(x(s))\right)^{\frac{1}{\alpha}}x'(s)ds\le\int_{T_2}^{t}\left(\frac{-C_{T_2}}{r(s)\rho(s)}\right)^{\frac{1}{\alpha}}ds,$$

$$\int_{x(T_2)}^{x(t)}k_2\left(\psi(y)\right)^{\frac{1}{\alpha}}dy\le\int_{T_2}^{t}\left(\frac{-C_{T_2}}{r(s)\rho(s)}\right)^{\frac{1}{\alpha}}ds.$$

From (2.3) and $0 < x(t) \le x(T_2)$, this implies that $\int_{x(T_2)}^{x(t)} k_2 \left(\psi(y)\right)^{\frac{1}{\alpha}} dy$ is lower bounded, but the right side of it tends to mines infinity. Then, this is a contradiction.

Example 2.2 Consider the following differential equation

$$\left[\frac{1}{t}(13x'(t) + \frac{x'(t)}{(x'(t))^2+1}\right]' + \left(t + \frac{\sin t}{t}\right)x(t) = \frac{2x^8 \sin t \cos(x'(t)+1)}{(x^7+1)t^3}, \quad t \ge \frac{\pi}{2},$$

Evidently, if we take $p(t) = \frac{2}{t^3}$, $\rho(t) = t$ and $\alpha = 2$. Then all conditions of Theorem 2.1 are satisfied, hence, all the solutions are oscillatory.

Theorem 2.3 If (A_1) – (A_3), conditions (2.1) – (2.3) hold, and

$$\int_{T_0}^{\infty} \rho(s)[q(s) - p(s)]ds < \infty , \tag{2.10}$$

$$\liminf_{t\to\infty} \left[\int_T^t Z(s)ds\right] \ge 0 \quad \text{for all large } T , \tag{2.11}$$

$$\lim_{t\to\infty} \int_{T_0}^{t} \left(\frac{1}{\rho(s)r(s)} \int_s^{\infty} Z(u)d\right)^{1/\alpha} ds = \infty, \tag{2.12}$$

and

$$\int_{\pm\varepsilon}^{\pm\infty} \left(\frac{\psi(y)}{g(y)}\right)^{\frac{1}{\alpha}} dy < \infty \quad \text{for every } \varepsilon > 0. \tag{2.13}$$

Thus all solutions of Eq. (1.1) are oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution on $[T,\infty)$, $T \ge T_0$ of Eq. (1.1). Let us assume that $x(t)$ is positive on $[T,\infty)$ and consider the following three cases for the behavior of $x'(t)$.

Case 1: $x'(t) > 0$ for $T_1 \ge T$ for some $t \ge T_1$; then from (2.10), we obtain

$$\int_{T_1}^{t} Z(s)ds \le \frac{r(T_1)\rho(T_1)\psi(x(T_1))\left|f(x'(T_1))\right|^{\alpha-1} f(x'(T_1))}{g(x(T_1))} - \frac{\rho(t)r(t)\psi(x(t))f(x'(t))^{\alpha}}{g(x(t))}.$$

From (2.1) and (2.2), we obtain

$$\frac{1}{r(t)\rho(t)}\int_{T_1}^{t} Z(s)ds \le \frac{\psi(x(t))\, f(x'(t))^{\alpha}}{g(x(t))}$$

Hence, for all $t \ge T_1$

$$\left(\frac{1}{r(t)\rho(t)}\int_{T_1}^{\infty} Z(s)ds\right)^{1/\alpha} \le \frac{\psi(x(t))^{1/\alpha}\, f(x'(t))}{g(x(t))^{1/\alpha}}$$

$$\int_{T_1}^{t}\left(\frac{1}{r(s)\rho(s)}\int_{s}^{\infty} Z(u)du\right)^{1/\alpha} ds \le k_1\int_{T_1}^{t} \frac{\psi(x(s))^{1/\alpha}\, x'(s)}{g(x(s))^{1/\alpha}}\, ds,$$

$$\le k_1 \int_{x(T_1)}^{\infty}\left(\frac{\psi(y)}{g(y)}\right)^{1/\alpha} dy.$$

Using (2.13), we obtain

$$\int_{T_1}^{t}\left(\frac{1}{r(s)\rho(s)}\int_{s}^{\infty} Z(u)du\right)^{1/\alpha} ds < \infty,$$

which contradicts to the condition (2.13)..

Case 2: If $x'(t)$ is oscillatory, then there exists a sequence $\{\alpha_n\} \to \infty$ on $[T,\infty)$ such that $x'(\alpha_n) < 0.$ Let us assume that N is sufficiently large so that

$$\int_{\alpha_N}^{\infty} Z(s)ds \ge 0.$$

Then, from (2.1), (2.2) and (2.7), we have

$$-C_{\alpha_N} - \int_{\alpha_N}^{t} Z(s)ds \ge -\frac{\rho(t)r(t)\psi(x(t))\left(-f(x'(t))\right)^{\alpha}}{g(x(t))}$$

$$C_{\alpha_N} + \int_{\alpha_N}^{t} Z(s)ds \le \frac{\rho(t)r(t)\psi(x(t))\left(-f(x'(t))\right)^{\alpha}}{g(x(t))}$$

Thus

$$\le \liminf_{t\to\infty} \frac{\rho(t)r(t)\psi(x(t))\left(-f(x'(t))\right)^{\alpha}}{g(x(t))} \ge C_{\alpha_N} + \liminf_{t\to\infty}\int_{\alpha_N}^{t} Z(s)ds > 0,$$

which contradicts to the assume that $x'(t)$ oscillates.

Case 3: Let $x'(t)<0$ for $t\geq T_1$. Condition (2.11) implies that for any $t_0\geq T_1$ such that

$$\int_t^{\infty}\rho(s)[q(s)-p(s)]ds\geq 0 \text{ for all } t\geq T_1.$$

The remaining part of the proof is similar to that of Theorem 2.1then will be omitted.

Example 2.4 Let us consider the following equation

$$\left[t\left(\frac{x^4(t)}{x^4(t)+1}\right)\left(7x'(t)+\frac{(x'(t))^5}{(x'(t))^4+1}\right)\right]'+\frac{1}{t^3}x^3(t)=\frac{x^3\cos x\sin 2x'(t)}{t^4},\ t>1,$$

Evidently, if we take $p(t)=\frac{1}{t^4}, \rho(t)=t$ and $\alpha=1$. Then the equation given in Example 2.2 is oscillatory by Theorem 2.2.

Remark 2.1

Condition (2.10) implies that $\int_T^{\infty}Z(s)\geq 0$ and $\liminf_{t\to\infty}\int_T^{\infty}Z(s)ds=\int_T^{\infty}Z(s)ds;$ hence (2.11) takes the form of $\int_T^{\infty}Z(s)\geq 0,$ for all large T.

Remark 2.2 when $\alpha=1, \psi(x(t))=1$ and $f(x'(t))=x'(t),$ Theorem 2.1 and 2.2 reduce to Theorem 1 and 2 Remili [27] and Theorems 2.1 and 2.3 are obtained by analogy with Theorems 2.1 and 2.2 from [31].

Theorem 2.5 Assume that

$$f(y)\geq by \quad \text{for all } y\in\mathbb{R} \text{ and for some constant } b>0, \tag{2.14}$$

$$0<\int_0^{\pm\varepsilon}\frac{\psi(u)}{g(u)}du<\infty \quad \text{for all } \varepsilon>0. \tag{2.15}$$

Furthermore, assume that there exist a constant A such that

$$\limsup_{t\to\infty}R(t)=A<\infty, \tag{2.16}$$

where $R(t)=\int_{t_0}^{t}\frac{ds}{r(s)}$, and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{1}{r(s)}\int_{t_0}^{s}[q(u)-p(u)]duds=\infty. \tag{2.17}$$

Then the differentia Eq. (1.2) is oscillatory.

Proof. Without loss of generality, let assume that there exists a solution $x(t)$ of (1.2) such that $x(t)>0$ on $[T,\infty)$ for some $T\geq t_0$. A similar argument holds also for the case when $x(t)<0$. Let $w(t)$ be defined by the Riccati Transformation

$$w(t)=\frac{r(t)\psi(x(t))f(x'(t))}{g(x(t))}\quad ,t\geq T.$$

Derivation this equality we have

$$w'(t)=\frac{\left(r(t)\psi(x(t))f(x'(t))\right)'}{g(x(t))}-\frac{r(t)\psi(x(t))f(x'(t))g'(x(t))x'(t)}{g^2(x(t))}.$$

This, and (1.2) imply

$$w'(t)\leq p(t)-q(t)\qquad t\geq T.$$

Integrating this inequality from T to $t(\geq T)$, we obtain

$$w(t)\leq w(T)-\int_{T}^{t}[q(s)-p(s)]ds$$

By condition (2.14), we get

$$\begin{aligned}b\frac{r(t)\psi(x(t))x'(t)}{g(x(t))}&\leq\frac{r(t)\psi(x(t))f(x'(t))}{g(x(t))}\\&\leq w(T)-\int_{T}^{t}[q(s)-p(s)]ds,\qquad b>0\end{aligned}$$

Integrating the above inequality multiplied by $\frac{1}{r(t)}$ from T to $t(\geq T)$, we have

$$b\int_T^t \frac{\psi(x(s))x'(s)}{g(x(s))}ds \le \int_T^t \frac{\psi(x(s))f(x'(s))}{g(x(s))}ds$$
$$\le w(T)R(t) - \int_T^t \frac{1}{r(s)} \int_T^s [q(u)-p(u)]duds.$$

From condition (2.16) and (2.17), we get that

$$\theta(t) = \int_T^t \frac{\psi(x(s))x'(s)}{g(x(s))}ds \to -\infty \text{ as } t \to \infty.$$

Now, if $x(t) \ge x(T)$ for large t then $\theta(t) \ge 0$, which is a contradiction. Hence for large t, $x(t) \le x(T)$, so

$$\theta(t) = -\int_{x(t)}^{x(T)} \frac{\psi(u)}{g(u)}du > -\int_0^{x(T)} \frac{\psi(u)}{g(u)}du > -\infty,$$

which is again a contradiction. This completes proof the Theorem 2.3.

Theorems 2.6 Suppose that conditions (2.14), (2.15) and (2.16) hold. Furthermore, suppose that, there exist a function $\rho:[t_0,\infty) \to (0,\infty)$ such that $\rho'(t) \ge 0$ for all $t \ge t_0$, and

$$\limsup_{t\to\infty} \int_{t_0}^t \frac{1}{\rho(s)r(s)} \left(\int_{t_0}^s \rho(u)[q(u)-p(u)]du \right) ds = \infty. \tag{2.18}$$

Then the differential equation (1.2) is oscillatory.

Proof. Without loss of generality, let assume that there exists a solution $x(t)$ of (1.2) such that $x(t) > 0$ on $[T,\infty)$ for some $T \ge t_0$. Let $w(t)$ be defined by the Riccati Transformation

$$w(t) = \rho(t) \frac{r(t)\psi(x(t))f(x'(t))}{g(x(t))} \quad ,t \ge T.$$

Derivation this equality we have

$$w'(t) = \frac{\rho(t)\left(r(t)\psi(x(t))f(x'(t))\right)'}{g(x(t))} + \frac{\rho'(t)r(t)\psi(x(t))f(x'(t))}{g(x(t))}$$
$$-\frac{\rho(t)r(t)\psi(x(t))f(x'(t))g'(x(t))x'(t)}{g^2(x(t))}.$$

This, and (1.2) imply

$$w'(t) \le -\rho(t)[q(t)-p(t)] + \frac{\rho'(t)}{\rho(t)} w(t).$$

Hence for all $t \ge T$, we obtain

$$\int_T^t \rho(s)[q(s)-p(s)]\,ds \le -\int_T^t \rho(s) \frac{d}{ds}\left(\frac{w(s)}{\rho(s)}\right) ds. \tag{2.19}$$

By the Bonnet's Theorem that for each $t \ge T$, there exist a $T_0 \in [T,t]$ such that

$$-\int_T^t \rho(s) \frac{d}{ds}\left(\frac{w(s)}{\rho(s)}\right) ds = -\rho(t)\int_{T_0}^t \left(\frac{d}{ds}\frac{w(s)}{\rho(s)}\right) ds = -\rho(t)\frac{w(t)}{\rho(t)} + \rho(t)\frac{w(T_0)}{\rho(T_0)}$$

$$-\int_T^t \rho(s) \frac{d}{ds}\left(\frac{w(s)}{\rho(s)}\right) ds = -w(t) + B\rho(t) \qquad ; B = \frac{w(T_0)}{\rho(T_0)}. \tag{2.20}$$

By (2.19) and (2.20) we get

$$\int_T^t \rho(s)[q(s)-p(s)]\,ds = -w(t) + B\rho(t). \tag{2.21}$$

Integrating the above inequality multiplied by $\dfrac{1}{r(t)\rho(t)}$ from T to $t (\ge T)$, we obtain

$$b\int_T^t \frac{\psi(x(s))x'(s)}{g(x(s))} ds \le \int_T^t \frac{\psi(x(s))\, f(x'(s))}{g(x(s))} ds$$
$$\le BR(t) - \int_T^t \left(\frac{1}{r(s)\rho(s)} \int_T^s \rho(u)[q(u)-p(u)]\,du \right) ds.$$

From (2.16) and (2.18), we have

$$\theta(t) = \int_T^t \frac{\psi(x(s))x'(s)}{g(x(s))} ds \to -\infty \text{ as } t \to \infty.$$

Now, if $x(t) \ge x(T)$ for large t, then $\theta(t) \ge 0$, which is a contradiction. Hence for large t, $x(t) \le x(T)$, so

$$\theta(t) = -\int_{x(t)}^{x(T)} \frac{\psi(u)}{g(u)} du > -\int_{0}^{x(T)} \frac{\psi(u)}{g(u)} du > -\infty,$$

which is again a contradiction. This completes proof the Theorem 2.6.

Example 2.7 Consider the differential equation

$$\left[e^{t} \frac{x^4(t)}{x^4(t)+1} x'(t) \right]' + \left(e^{2t} + \sin t \right) x^3(t) = \frac{x^7(t)\sin t}{(1+x^4(t))^2} \frac{(x'(t))^2}{(x'(t))^2+1}, \qquad t \geq 0.$$

Here,

$$r(t) = e^{t}, q(t) = e^{2t} + \sin t, \psi(x(t)) = \frac{x^4(t)}{x^4(t)+1}, g(x) = x^3, H(t, x(t), x'(t)) = \frac{x^7(t)\sin t}{(1+x^4(t))^2} \frac{(x'(t))^2}{(x'(t))^2+1},$$

$$\frac{H(t, x(t), x'(t))}{g(x)} = \frac{x^7(t)\sin t}{(1+x^4(t))^2} \frac{(x'(t))^2}{(x'(t))^2+1} \times \frac{1}{x^3} \leq \sin t = p(t).$$

So, can note that

$$\limsup_{t\to\infty} R(t) = \limsup_{t\to\infty} \int_{t_0}^{t} \frac{ds}{e^{s}} < \infty,$$

$$\int_{0}^{\pm\varepsilon} \frac{u}{(u^2)^2+1} du = \frac{1}{2} \tan^{-1} \varepsilon^2 < \infty.$$

Let us take $\rho(t) = 1$ we have

$$\int_{t_0}^{t} \left(\frac{1}{r(s)\rho(s)} \int_{t_0}^{s} \rho(u)[q(u) - p(u)] du \right) ds = \int_{T}^{t} \left(\frac{1}{e^{s}} \int_{T}^{s} [e^{2u} + \sin u - \sin u] du \right) ds = \infty,$$

then, Theorem 2.4 ensures that every solution of the equation given oscillates.

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