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Review on the Beauty and Interplay Chebyshev Polynomials: Some Properties and Identities of the First Four Kinds

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ABSTRACT

This paper present a sysmatic treatment of the four classical kinds of Chebyshev polynomials within a common analytic form. Although the first two kinds are widely recognised and routinly applied in approximation theory and numericall analysis, the third and fourth kinds have received comparatively less attention, despite their distintive properties. We consolidate and extends known results, derive some identities, and present new generating functions, both ordinary and exponential that reveal deeper connection among these families. Additionally, we provide systematic derivations of monomial expansion, and explicit formulas for derivatives. A unified framework for the interconnecting identities among all four kinds is established, facilitating their application in spectral methods and numerical analysis.

1. Introduction

Chebyshev polynomials constitute a remarkable class of orthogonal polynomials which serve a crucial role in many areas of mathematics, particularly in approximation theory, numerical analysis, and engineering. Named after the influential Russian mathematician Pafnuty Chebyshev, these polynomials exhibit fascinating properties that enhance their significance and utility. Chebyshev polynomials constitute a cornerstone of applied mathematics, renowned for their minimax properties.

Their preeminence stems from the Chebyshev approximation theorem, which guarantees that approximations based on these polynomials minimize the maximum error. This property makes them as the gold standard in numerical methods such as interpolation, quadrature, and spectral methods, ensuring rapid convergence and numerical stability (Trefethen, 2019).

The inherent beauty of Chebyshev polynomials lies in their dual nature, they are algebraic constructs defined by recurrence relations, yet they possess direct trigonometric representations. This duality emphasizes the connection between algebra, analysis and geometry, enriching their theory and application.

There are many types of Chebyshev polynomials, but the more known ones remain the first T_n and the second U_n kinds. By contrast, the third $V_n$and the fourth W_n kind have received less attention despite their significant utility. Recent work has begun to explore these lesser-known types, revealing a wealth of interconnected identities.

The connection between Chebyshev polynomials and trigonometric functions highlights a profound unity within mathematics, bridging algebra, geometry, and analysis.

Chebyshev polynomials have been extensively studied, leading to a rich body of knowledge regarding their properties and applications. Mason and Handscomb (2003) provide a comprehensive overview, discussing their significance in approximation theory and presenting a unified treatment of the first four kinds. Rivlin (1974) explores orthogonality and recurrence relations, emphasizing numerical analysis applications,

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while Szegö (1939) places Chebyshev polynomials within the broader theory of orthogonal polynomials, establishing rigorous analytical foundations. Mason (1993) highlights the applications of the U_n , V_n and W_n kinds in integral transforms, demonstrating that these less-studied kinds possess unique properties complementary to the first two.

Gautschi (2004) focuses on the computation and approximation of orthogonal polynomials, providing practical algorithms for numerical methods. Recent contributions have expanded the field: Abd-Elhameed (2021) delves into the third and fourth kinds, providing novel expressions and identities, and also introduces a sixth kind for spectral solutions of nonlinear PDEs; Doha and Abd-Elhameed (2014) investigate integrals of the V_n and W_n kinds, applying them to problems in boundary value with polynomial coefficients; Bedratyuk and Lunio (2019, 2022) derive derivations and identities for the first and second kinds, establishing systematic algebraic frameworks; Kishore and Verma (2023) explore connections between Chebyshev polynomials, Fibonacci polynomials, and their derivatives; Verma et al. (2024) characterize the third and fourth kinds through their characteristic equations; and Qi, Niu, and Lim (2019) provide explicit and inversion formulas for the first two kinds. Boyd (2001) and Trefethen (2000, 2019) offer extensive treatments of Chebyshev spectral methods, bridging theory and computation, while Boyd and Petschek (2014) examine the connection among Chebyshev, Legendree, and Jacobi polynomials, establishing conditions for the generic advantages of Chebyshev polynomials. Despite this extensive body of work, a unified treatment of all four kinds together with a newly defined fifth kind is lacking. The present paper addresses this gap by synthesizing known results, deriving new identities.

This paper is structured into the following manner. Section 2 review some essential preliminaries. Section 3 define the four Chebyshev kinds via Jacobi polynomials, recurrence formula, orthogonality, generating function and Rodrigues formula. Section 4 presents and derives presents a comprehensive set of algebraic identities, connection forms, and derivative relations. Section 5 gives explicit series representations, including forms for even and odd indices. Section 6 provides monomial expansions and inversion formulas.

2. Preliminary

We begin by recalling some important concept used throughout of this work:
For an integer $n > 0$, the rising factorial also known as called the Pochhammer symbol $(\alpha)_s$ is

$$(\alpha)_{s.} = \alpha.(\alpha+1))\ \ldots.(\alpha+s-1), \quad (\alpha)_0 = 1$$

Equivalently

$$(\alpha)_{s.} = \frac{\Gamma(\alpha+s)}{\Gamma(\alpha)}$$

A related identity

$$(\alpha)_{s.} = (-1)^s(1-a-s)_s$$

Gamma function $\Gamma(\alpha)$ for $\alpha > 0$ is given by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt\,, \qquad \text{(Euler integral)}$$

It satisfies $\Gamma(\alpha.+1) = \alpha.\Gamma(\alpha.)$ and is related to the Pochhammer symbol via
For special value

$$\Gamma\left(s.+\frac{1}{2}\right) = \prod_{m.=0}^{n.-1}\left(s.-m.-\frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)_s.\sqrt{\pi}$$
$$= \frac{(2s.-1)!!}{2^s}\sqrt{\pi}$$

$$\Gamma\left(s+\frac{3}{2}\right) = \prod_{m=0}^{s-1}\left(s-m+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \left(\frac{3}{2}\right)_s\frac{\sqrt{\pi}}{2}$$
$$= \frac{(2s+1)!!}{2^{s+1}}\sqrt{\pi}$$

For integer m,n and real a, with $n \le a$, the following useful form holds:

$$(-a)_s = \frac{(-1)^s a!}{(a-s)!}, \quad \frac{(-s)_m}{s!} = \frac{(-1)^m}{(s-m)!} \ ,$$
$$(a)_{2s} = 2^{2s}\left(\frac{a}{2}\right)_s\left(\frac{a+1}{2}\right)_s.$$

Also

$$(a)_{n-m} = \frac{(-1)^m(a)_n}{(1-a-n)_m}\ , \quad (a)_{n+m} = (a)_m(a+m)_n$$

For positive large values of n, Stirling's formula provides the asymptotic approximation

$$\Gamma(s.+1) \sim \sqrt{2s.\pi}\ \left(\frac{s.}{e}\right)^{s.}, \qquad n >\gg 1,$$

For a real number $\alpha,\beta > 0$,the Beta functions $B.(\alpha.,\beta.)$

$$B.(\alpha.,\beta.) = \int_0^1 t^{\alpha.-1}.(1-t)^{\beta.-1}.dt = \frac{\Gamma(\alpha.).\Gamma(\beta.)}{\Gamma(\alpha.+\beta.)}$$

The four classical kinds are specific realizations of the Jacobii polynomials family $P_n^{(\alpha,\beta.)}(x)$ that are orthogonal over $[-1.,1]$

$$w(x) = (1-x)^{\alpha.}.(1+x)^{\beta.}, \quad \alpha,\beta > -1$$

Case $\alpha = -\frac{1}{2}, \beta = \frac{1}{2}$

$$P_{n.}^{\left(-\frac{1.}{2},\frac{1}{2}\right)}(\cos\theta.) =.\frac{\left(\frac{1.}{2}\right)_{n.}}{n!.}\frac{\cos\left[.\left(\frac{2n.+1}{2}\right).\theta\right]}{\cos\left(\frac{\theta.}{2}\right)}.$$

At $x = 1$, this simplifies to

$$P_n^{\left(.-\frac{1}{2},\frac{1}{2}\right)}(1) = \frac{\left(\frac{1}{2}\right)_{n.}}{n!.}, \quad where \ \left(\frac{1}{2}\right)_{n.} = \frac{(2n.-1)!!.}{2^{n.}}$$

Case $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$

$$P_{n.}^{\left(\frac{1}{2},-\frac{1}{2}\right)}(1) = \frac{(2n+1)\left(\frac{1}{2}\right)_{n.}}{n!}.$$

Consequently, the normalized form is

$$\frac{P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}.(\cos\theta.)}{P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(1)}.=.\frac{\cos\left[\left(n.+\frac{1}{2}\right)\theta.\right]}{\cos\left(\frac{\theta}{2}\right)}.$$

Jacobi polynomials satisfy:

$$P_{n.}^{(\alpha.,\beta.)}(-1) =.(-1)^{n.}.P_{n.}^{(\beta,\alpha.)}(1).$$
$$P_{n.}^{(\alpha.,\beta.)}(-x) =.(-1)^{n.}.P_{n.}^{(\beta.,\alpha.)}(x).$$

For large n, the asymptotic behavior at $x = .1$ is

$$P_{n.}^{(\alpha.,\beta.)}(1) \sim \frac{\sqrt{2\pi.(n+\alpha)}.\left(\frac{n.+\alpha}{e}\right)^{n.+\alpha}}{\alpha!.\sqrt{2\pi n.}.\left(\frac{n.}{e}\right)^{n.}} \sim n.^{\alpha}, \quad n. >\gg 1.$$

3. Chebyshev polynomials Kinds

The first four kinds Chebyshevv $T_{n.}, U_{n.}, V_{n.}$ and $W_{n.}$ are obtained by the $P_{n.}^{(\alpha,\beta.)}(x)$ for special values α and β. (Doha & Abd-alhameed, 2014)

First kind ($\alpha = \beta = -\frac{1}{2}$)

$$T_{n.}(\mathrm{x}) = \cos(n.\theta) = \frac{P_n^{\left(-\frac{1.}{2},-\frac{1.}{2}\right).}(cos\,\theta)}{P_{n.}^{\left(-\frac{1.}{2},-\frac{1}{2}\right)}(1)}$$
$$= \frac{P_{n.}^{\left(-\frac{1.}{2},-\frac{1}{2}\right)}(\mathrm{x}).n!.}{\left(\frac{1}{2}\right)_{n.}} \qquad (1)$$

where

$$P_{n.}^{\left(-\frac{1.}{2},-\frac{1.}{2}\right)}(1) = \frac{\left(\frac{1}{2}\right)_{n.}}{n!..}.$$

Second kind ($\alpha = \beta = \frac{1}{2}$)

$$U_{n.}(\mathrm{x}) = \frac{\sin[.(n.+1)\theta.]}{\sin(\theta)} = \frac{(n.+1)}{P_{n.}^{\left(\frac{1.}{2},\frac{1}{2}\right)}(1)}.P_{n.}^{\left(\frac{1}{2},\frac{1}{2}\right)}(cos\,\theta)$$
$$= \frac{(n.+1)}{\binom{n.+\frac{1}{2}}{n..}}P_{n.}^{\left(\frac{1.}{2},\frac{1}{2}\right)}(\mathrm{x}) \qquad (2)$$

where

$$P_{n.}^{\left(\frac{1}{2},\frac{1}{2}\right)}(1) = \frac{\left(\frac{3}{2}\right)_{n.}}{n!.}, \quad \left(\frac{3}{2}\right)_{n.} = \frac{(2n.+1)!!}{2^{n.}}$$

Therefore

$$\frac{(n.+1).}{P_{n.}^{\left(\frac{1.}{2},\frac{1}{2}\right)}(1).} =.\frac{(n.+1)!..2^{n}.n!.}{(2n.+1)!.}.$$

and

$$V_{n.}(x) = \frac{cos\left(n.+\frac{1}{2}\right)\theta}{cos\left(\frac{1}{2}\right)\theta} = 2^{2n.}.\frac{P_{n.}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x)}{\binom{2n.}{n.}} \qquad (3)$$

$$W_{n.}(x) = \frac{\sin(n.+\frac{1}{2})\theta}{\sin(\frac{1}{2}\theta)}$$
$$= 2^{2n.}.\frac{P_n^{\left(\frac{1.}{2},-\frac{1}{2}\right)}(x)}{\binom{2n.}{n.}} \qquad (4)$$

the same three-term remarkable recurrence relations hold for all kinds: (Mason & Handscomb, 2003).

$$P_n(x) = 2xP_{n-1} - P_{n-2}(x), \quad n \geq 2 \qquad (5)$$

differing only in their initial conditions:

kind	$P_0(x)$	$P_1(x)$
$T_{\mathrm{n.}}$	1	x
$U_{\mathrm{n.}}$	1	$2x$
$V_{\mathrm{n.}}$	1	$2x-1$
$W_{\mathrm{n.}}$	1	$2x$+1

Each kind solution to differential equations:

$T_{n.}: (1-x^2).y''. - x.y'. + n.^2.y = 0$
$U_{n.}: (1-x^2).y''. - 3.x.y' + n(n+2)y = 0 \quad (6)$
$V_{n.}: (1-x^2).y'' + (1-2x).y' + n.(n.+1).y = 0$
$W_{n.}: (1-x^2).y'' + (1+2x).y' + n.(n.+1).y = 0$

The zeros of these polynomials are explicitly known and play a crucial for numerical methods like Gaussian quadrature. For $j = 1,2,\ldots,n$.

$$T_{n.}(x_j) = 0 \Longrightarrow \ x_j = \cos\left[\frac{(2j-1)\pi}{(2n.)}\right]$$
$$U_{n.}(x_j) = 0 \Longrightarrow \ x_j = \cos\left[\frac{j\pi}{(n.)}\right]$$
$$V_{n.}(x_j) = 0 \Longrightarrow \ x_j = \cos\left[\frac{(2j-1)\pi}{(2n.+1)}\right],$$
$$W_{n.}(x_j) = 0 \Longrightarrow \ x_j = \cos\left[\frac{(2j)\pi}{(2n.+1)}\right] \qquad (7)$$

Among monic polynomials $\frac{1}{2^{n.-1}}T_{n.}$, $\frac{1}{2^{n.}}U_{n.}$, $\frac{1}{2^{n.}}V_{n.}$, $\frac{1}{2^{n.}}W_{n.}$ minimize the Chebyshev norm on $[-1,1]$ under with the respective of weight 1, $(1-x^2)^{1./2}$, $(1+x.)^{1./2}$, $(1-x.)^{1./2}$

Orthogonality over $.[-1,1]$ in respect to distinct weight functions (Mason & Handscomb, 2003).

$\int_{-1}^{1} T_{n.}\,.T_{m}..\frac{1.}{\sqrt{1-x^{2}.}}dx=\frac{\pi}{2.}\delta_{n.m.},$

$\int_{-1}^{1} U_{n.}\,.U_{m.}\,.\sqrt{1-x^{2.}}dx=\frac{\pi}{2}\delta_{n.m.}$

$\int_{-1}^{1} V_{n.}\,.V_{m.}\,.\frac{\sqrt{1+x}}{.\sqrt{1-x}}.dx=\pi.\delta_{n.m.},$

$\int_{-1}^{1} W_{n.}\,.W_{m.}\,.\frac{\sqrt{1-x}}{\sqrt{1+x}.}dx=\pi.\delta_{n.m.}$

The orthogonality of Chebyshev polynomials allows them to serve as excellent bases for expanding more complex functions. If f defined over $[-1,1]$ satisfies certain continuity conditions (e,g., Lipschitz Continuity), it could expanded within a uniformly and absolutely convergent series for each kind (Trefethen, 2019):

$$f(x)=\frac{a_{0.}}{2}+\sum_{k=0}^{\infty}a_{k.}\,.T_{k},\qquad a_{k.}=\frac{2}{\pi}.\int_{-1}^{1}f(x).T_{k}.\frac{.dx}{\sqrt{1-x^{2}}}$$

$$f(x)=\frac{b_{0.}}{2}+\sum_{k=1}^{\infty}b_{k.}\,.U_{k}\ ,$$

$$b_{k.}\frac{2}{\pi}.\int_{-1}^{1}f(x).U_{k}\,.\sqrt{1-x^{2}}.dx$$

$$f(x)=\frac{c_{0.}}{2}+\sum_{k=1}^{\infty}c_{k}\,.V_{k.}\ ,$$

$$c_{k.}=\frac{2}{\pi}.\int_{-1}^{1}f(x)\,.V_{k.}\sqrt{\frac{1+x}{1-x}}.dx$$

$$f(x)=\frac{d_{0.}}{2}+\sum_{k=1}^{\infty}d_{k.}\,.W_{k}\ ,d_{k}=\frac{2}{\pi}.\int_{-1}^{1}f(x).W_{k.}\sqrt{\frac{1-x}{1+x}}dx \tag{8}$$

3.1 The generating Function

Generating functions provide a powerful way to encapsulate sequences of polynomials into a single analytic expression. For Chebyshev polynomials, these functions not only offer alternative definitions but also serve as foundations for deriving identities, connection formulas, and asymptotic properties. The ordinary generating functions for the four primary kinds of Chebyshev polynomials are remarkably similar, differing only in their numerator, which dictates their specific initial conditions and properties (Mason & Handscomb, 2003).

The generating function provide a power series representation for each sequence, $|x|<1,|t|<1$:

$$\sum_{n=0}^{\infty}T_{n}..t^{n.}=\frac{1-t.x}{1-2t.x+t^{2}}$$

$$\sum_{n=0}^{\infty}U_{n.}\,.t^{n.}=\frac{1}{1-2x.t+t^{2}}$$

$$\sum_{n=0}^{\infty}V_{n.}.t^{n.}=\frac{1-t}{1-2t.x+t^{2}}, \tag{9}$$

$$\sum_{n=0}^{\infty}W_{n}..t^{n.}=\frac{1+t}{1-2x.t+t^{2}}$$

These can be derived directly from the trigonometric definitions of the polynomials by recognizing the resulting series as geometric in form. For instant, the first kind:

$$\sum_{n.=0}^{\infty}T_{n.}(x).t^{n.}=\sum_{n=0}^{\infty}\cos.\theta\mathrm{n}.t^{n.}$$

$$=eal\Re\left(\sum_{n=0}^{\infty}\mathrm{e}^{in.\theta}.t^{n.}\right)$$

$$=eal\Re.\left(\sum_{n.=0}^{\infty}\left(\mathrm{e}^{i\theta}\mathrm{t}\right)^{n.}\right)$$

$$=eal\Re\left(\frac{1}{1-t.\mathrm{e}^{i\theta}}\right)$$

$$\left(.\frac{1.}{1-t.\mathrm{e}^{i\theta}}\right)=\frac{1-t\mathrm{e}^{-i\theta}}{1-t2.\cos.\theta+t^{2}}$$

$$=\frac{1-\theta\mathrm{cos}t+\mathrm{it}.\theta\mathrm{sin}}{1-x2.t+t^{2}}$$

Taking the real part gives $\frac{1-tx}{1-2tx+t^{2}}$. The other formulas follow similarly using the trigonometric definitions and geometric series.

Beyond the standard generating functions, exponential generating functions lead to elegant closed-form expressions and reveal deeper connections to transcendental functions, $|x|<1,\ |t.|<1$:

$$\sum_{k.=0}^{\infty}\frac{T_{k.}\,.t^{k.}}{k!.}=e^{t..x.}cos\left(t.\sqrt{1-x^{2}}\right)$$

$$\sum_{k.=1}^{\infty}\frac{U_{k.}}{k!.}t^{k.}=e^{t.x}\frac{sin(\sqrt{1-x^{2}})}{\sqrt{1-x^{2}}} \tag{10}$$

$\sum_{k=1}^{\infty}\frac{V_{k.}}{k!.}t^{k}=e^{t.x}\,sec\left(\frac{t}{2}\right).cos\left(t\sqrt{1-x^{2}}+\frac{t}{2}\right)$

$$\sum_{k.=1}^{\infty}\frac{W_{k.}}{k!.}t^{k.}=e^{t.x}.sec\left(\frac{t}{2}\right).sin\left(t.\sqrt{1-x^{2}}+\frac{t}{2}\right)$$

To prove the first one:

$$\sum_{k.=0}^{\infty}\frac{T_{k.}\,.\,t^{k.}}{k!.}=\Re eal\sum_{k=0}^{\infty}\frac{1}{k!.}.\left(te^{i\,arccos(x)}\right)^{k.}$$
$$=\Re eal\left(e^{te^{i\,arccos(x)}}\right).$$

Let $\theta=\arccos x$. Then we have

$$\Re eal\left(e^{te^{i\theta}}\right)=\Re eal\left(e^{t(\cos\theta+i\sin\theta)}\right)=e^{t\cos\theta}\Re eal\left(e^{ti\sin\theta}\right)$$

Where

$$\Re eal\left(e^{ti\sin\theta}\right)=\Re eal(cos(t.\sin\theta)+i\,sin(t\sin\theta)=cos(t\sin\theta).$$

Then, we get the result. For V_k

$$\sum_{k.=1}^{\infty}\frac{V_{k.}}{k!.}t^{k.}=\frac{1}{cos\left(\frac{\theta}{2}\right)}\Re eal\sum_{K=0}^{\infty}\frac{\left(e^{i\frac{\theta}{2}}.e^{ik\theta}\right)}{k!}$$
$$=\frac{1}{cos\left(\frac{\theta}{2}\right)}\Re eal\left(e^{te^{i\frac{\theta}{2}}}.e^{-i\frac{\theta}{2}}\right)$$

After factoring $e^{i\frac{\theta}{2}}$ or alternatively, using identity

$$\mathrm{V}_{\mathrm{n.}}=\mathrm{U}_{\mathrm{n.}}-\mathrm{U}_{\mathrm{n.}-1}\,.$$

and generating function for U_{n} .

Let a=$e^{i\frac{\theta}{2}}=\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}$

$$\sum_{k=1}^{\infty}\frac{V_k}{k!}t^k=\frac{1}{cos}\Re\sum_{K=0}^{\infty}\frac{(ta)^n a}{k!}=\frac{a}{cos\left(\frac{\theta}{2}\right)}\Re\left(e^{ta}\right)$$

Then, $a^{ta}=a^{t\cos\left(\frac{\theta}{2}\right)}a^{it\sin\left(\frac{\theta}{2}\right)}$, multiplying by a and taking the real part yields after simplification.

Following a similar methodology, one can derive exponential generating functions for others. These forms are equivalent and can be transformed into one another using trigonometric identities. The presence of the secant term indicates a more complex relationship between the fourth kind and the exponential function, likely arising from its definition involving $\sin\frac{\theta}{2}$ in the denominator. These results are non-trivial. A complete proof would typically start from the trigonometric definition of $W_n(x)$, express the sum using complex exponentials, and carefully combine the series to isolate the reals and imaginary parts that yield cosine and sine expressions above.

A further relationship between the second and another kind can be found by manipulating their generating functions:

Proposition 3.1

$$-\frac{1}{2}ln(1-2x.t+t^2)=\sum_{n=1}^{\infty}\frac{T_{n.}}{n.}.t^{n.} \qquad (11)$$

Proof: -We start from ordinary generating function of T_n.

$$\frac{1}{t.}.\left(\frac{1-t.x}{1-2x.t+t^2}-1\right)=\frac{x-t}{1-2x.t+t^2}.$$
$$=\sum_{n.=1}^{\infty}T_{n.}(x)\,.\,t^{n.-1}$$

Integrating in respec to t, we obtain

$$-\frac{1}{2}ln(1-2x.t+t^2)=\sum_{n=1}^{\infty}\frac{T_{n.}}{n.}.t^{n.}+C.$$

Letting $t=0$, we have $-\frac{1}{2}ln(1)=0+C$. Hence $C=0$, so we get the result.

Proposition 3.2

$$U_{n.}=\frac{1}{n+1}\sum_{m.=0}^{n.}T_{m.}\,.\,U_{n.-m.}\,. \qquad (12)$$

Proof: Let start with the

$$\frac{1}{1-2x.t+t^2}=\sum_{n.=0}^{\infty}U_{n.}.t^{n.}$$

Differentiating in respect to t, yield

$$2(x-t).(1-2x.t+t^2)^{-2}=\sum_{n.=0}^{\infty}n.U_{n.}\,t^{n.-1}$$

Multiplying by t, we get

$$(2x.t-2t^2).(1-2x.t+t^2)^{-2}=\sum_{n.==0}^{\infty}n.U_{n.}\,t^{n.}$$

Therefore

$$\frac{(2xt-2t^2)+(1-2x.t+t^2)}{(1-2x.t+t^2)^2}=\frac{1-t^2}{(1-2x.t+t^2)^2}$$
$$=\sum_{n.=0}^{\infty}(n.+1)U_n.t^{n.}$$

We can see that

$$\frac{1-t^2}{(1-2x.t+t^2)^2}=\frac{1-t^2}{(1-2x.t+t^2)}\frac{1}{(1-2x.t+t^2)}$$

Therefore, by definition of generating function of both $T_n\ and U_n$, we have

$$\frac{1-t^2}{(1-2x.t+t^2)}\frac{1}{(1-2x.t+t^2)}$$

$$=\sum_{n.=0}^{\infty}\left(\sum_{m.=0}^{n.}T_{m.}\,.\,U_{n.-m.}\right).t^{n.}$$

This demonstrates how generating function manipulation can be a potent tool for discovering and proving relationships among different families of polynomials.

3.2 The Rodrigues Formula

From the definition of Jacobi polynomials for $x\in(-1,1)$, the Rodrigues type as:

$$P_{n.}^{(\alpha..,\beta)}(x)=\frac{(-1)^{n.}}{2^{n.}.n!.}.(1-x)^{-\alpha}.(1+x)^{-\beta}.\frac{d^{n.}}{dx^{n}.}\left[(1-x)^{n.+\alpha}.(1+x)^{n.+\beta}\right].$$

By setting the parameters appropriately, we obtain Rodrigues formula each kind:

$$T_{n.} = \frac{(-1)^{n.}}{2^{n}n!.}.\sqrt{1-x^2}.\times\left(\frac{d}{dx}\right)^{n.}.(1-x^2)^{n.-\frac{1}{2}} ,$$

$$U_{\text{n.}} = \frac{(-1)^{n.}.(n.+1).\sqrt{\pi}}{2^{n.+1.}\left(n.+\frac{1}{2}\right)!}(1-x^2)^{-\frac{1}{2}}$$

$$\times\left(\frac{d}{dx}\right)^{n}(1-x^2)^{n+1/2} .$$

$$V_{n.} = 2^{n.}.\frac{(-1)^{n.}.(n)!.}{(2n.)!.}.\sqrt{\frac{1-x}{1+x}}\times\frac{d^{n.}}{dx^{n.}}.\left\{(1-x^2)^{n.}.\sqrt{\frac{1+x}{1-x}}\right\}$$

$$W_{n.} = 2^{n.}.\frac{(-1)^{n.}.(n.)!.}{(2n.)}.\sqrt{\frac{1+x}{1-x}}\times\frac{d^{n.}}{dx^{n.}}.\left\{(1-x^2)^{n.}.\sqrt{\frac{1-x}{1+x}}\right\}$$

(13)

3.3 Some of Derivative Formulas

Jacobi polynomials satisfy a simple derivative formula

$$\frac{d}{dx}P_{n.}^{(\alpha.,\beta.)} = \frac{1}{2}.(1+\alpha+\beta+n).P_{n-1}^{(\alpha+1,\beta+1)}(x)$$

Where the k-th derivative for $0 < k \le n$ (Bedratyuk & Luno, 2019) and (Bedratyuk & Luno, 2019)

$$\frac{d}{dx}^{k}P_{n.}^{(\alpha,\beta)} = 2^{-k}\frac{\Gamma(1+\alpha+\beta+n+k)}{(1+\alpha+\beta+n)}_{k}P_{n-k}^{(\alpha+k,\beta+k)}(x).$$

Applying the above with the appropriate parameters yields:

$$\frac{d}{dx}^{k}T_{n.}(\text{x}) = 2^{-k.}.\frac{(n.+\text{k}.-1)!.n.}{\left(\frac{1}{2}\right)_{n.}}.P_{n.-k}^{\left(k-\frac{1}{2},\text{k}-\frac{1}{2}\right)}(\text{x})$$

Second kind

$$\frac{d}{dx}^{k}U_{n.}(\text{x}) = 2^{-k.}.\frac{(n.+\text{k}.+1)!}{n!.\binom{n.+\frac{1}{2}}{n.}}.P_{n.-k.}^{\left(k+\frac{1}{2},\text{k}+\frac{1}{2}\right)}(\text{x})$$

(14)

The following derivative express k-derivative of each family as a finite series:

$$\frac{d^k}{dx^k}T_{n.} = \sum_{m.=0}^{\left[\frac{n.-k-1}{2}\right]}.\frac{2^{k-1}.(n.-m-1)!.(m.+k-1).}{m!.(k-1)!.(n.-m-k)!.}.T_{n.-2m-k}$$

$$\frac{d^k}{dx^k}U_{n.} = \sum_{m=0}^{\left[\frac{n.-k-1}{2}\right]}\frac{2^{k.}n.(n-m-1)!(m+k-1)!.}{.m!(k-1)!.(n-m-k+1)!}U_{n-2m-k}$$

$$\frac{d^k}{dx^k}V_{n.} = \sum_{m=0}^{\left[\frac{n.-k}{2}\right]}\frac{2^{k.}(n.-m)!.(m+k-1)!.}{.m!.(k-1)!.(n-m-k)!}V_{n.-2m-k} + \sum_{m.=0}^{\left[\frac{n.-k-1}{2}\right]}\frac{2^{k}(n.-m-1)!.(m+k)!.}{m!.(k-1)!(n.-m-k)!}V_{n.-2m-k-1}$$

$$\frac{d^k}{dx^k}W_{n.} = \sum_{m=0}^{\left[\frac{n.-k}{2}\right]}\frac{2^{k}(n.-m)!.(m+k-1)!.}{.m!.(k-1)!.(n-m-k)!}W_{n.-2m-k} - \sum_{m.=0}^{\left[\frac{n.-k-1}{2}\right]}\frac{2^{k}(n.-m-1)!.(m+k)!}{m!.(k-1)!.(n.-m-k)!.}W_{n.-2m-k-1}$$

The formula for third and fourth can be rewritten in an equivalent way

$$\frac{d^k}{dx^k}V_{n.} = \sum_{i=0}^{n.-k}E_{n.,i,k}V_i$$

$$\frac{d^k}{dx^k}W_{n.} = \sum_{i=0}^{n.-k}(-1)^{n.+i+k}E_{n.,i,k}W_i ,$$

Where the coefficients

For even $(n.-i-k.)$

$$E_{n.,i,k} = \frac{2^k}{(k.-1)!.}\frac{\left(\frac{n.-i+k.-2}{2.}\right)!.\left(\frac{n.+i+k.}{2.}\right)!.}{\left(\frac{n.-i-k.}{2.}\right)!.\left(\frac{n.+i-k.}{2.}\right)!.}$$

For odd $(n.-i-k)$

$$E_{n.,i,k} = \frac{2^k}{(k.-1)!}\frac{\left(\frac{n.-i+k.-1}{2}\right)!\left(\frac{n.+i+k.-1}{2}\right)!}{\left(\frac{n.-i-k.-1}{2}\right)!\left(\frac{n.+i-k.+1}{2}\right)!}$$

From the definition and $\text{V}_{\text{n.}} = \text{U}_{\text{n.}} - \text{U}_{\text{n.}-1}$, while those $W_{n.}(x)$ are obtained by $W_{n.}(x) = (-1)^{\boldsymbol{n.}}V_{n.}(-x)$. See also (Doha et al., 2015).

3.4 Indefinite Integrals representations

From the trigonometric definitions and recurrence relations (Mason & Handscomb, 2003), we obtain

$$\int T_{n.}.dx = \frac{1}{2}.\left(\frac{T_{n.+1}}{n.+1}-\frac{T_{n.-1}}{n.-1}\right)+C$$

$$\int U_{n.}.dx = \frac{T_{n.+1}}{n.+1}+C$$

$$\int V_{n.}.dx = \frac{T_{n.+1}}{n.+1}-\frac{T_{n.}}{n.}+C \qquad (15)$$

$$\int W_{n.}.dx = \frac{T_{n..+1}}{n.+1}+\frac{T_{n.}}{n.}+C$$

The proof from the definitions. Also, for V_n and W_n can be proved by the relations of

$$V_n = U_n - U_{n-1} , W_n = U_n + U_{n-1}$$

respectively.

4. A Web of Interconnecting Identities

A particularly elegant feature of Chebyshev polynomials is the intricate web of the identities that connects the various kinds. These relationships often reflect the trigonometric identities from which the polynomials are derived. Consequently, reduces each polynomial identity to well-known trigonometric formula. Some of these relations are known and some not based on the researcher knowledge. See also (Mason & Handscomb, 2003), (Doha & Abd-alhameed, 2014), (Bedratyuk&Lunio, 2019), and (Verma et al., 2024):

$$T_{n.} = \frac{1}{2}[U_{n.} - U_{n.-2}] = U_{n.} - xU_{n.-1}$$

$$T_{n.+1} = xT_{n.} - (1-x^2)U_{n.-1}$$

$$\mathrm{V_{n.} = U_{n.} - U_{n.-1}\,,\ \ W_{n.} = U_{n.} + U_{n.-1}}$$

$$2\mathrm{T_{n.}}. = V_{n.} + V_{n.-1} = W_{n.} - W_{n.-1}$$

$$U_{n.} = \tfrac{1}{2}[V_{n.} + W._{n}],\ \ U_{n.} = \tfrac{1}{2(x-1)}[W_{n.} - W_{n.-1}]$$

$$T_{n.} - T_{n.-1} = (\mathrm{x}-1)W_{n.-1}$$

$$T_{n}. + T_{n.-1} = (\mathrm{x}+1)V_{n.-1}$$

$$W_{n.}(x) = (-1)^{\boldsymbol{n.}}.V_{n.}(-x), \qquad V_{n.}(-x) = (-1)^{n.}.W_{n.}(x)$$

The identities are proved by straightforward trigonometric manipulations or by using the definitions with $x = \ cos\theta$. For example

$$\cos(n.+1)\theta. = \cos.n.\theta.cos\theta - \sin.n.\theta\,.sin\theta$$
$$= \cos.n.\theta\,.cos\theta - \frac{\sin.n.\theta\ .}{sin\theta.}sin^2\theta$$

These relations can be proved by trigonometric identities.

$$\cos(n.)\theta - \cos.(n.-1)\theta = -2.sin\left(n.-\frac{1}{2}\right).sin\left(\frac{\theta}{2.}\right)$$

Some Product-to-sum formulas are particularly striking because of enabling the simplification of products into sums of single polynomials:

$$T_{n.}.T_{m.} = \frac{1}{2}.[T_{n.+m.} + T_{n.-m}.]$$

$$T_{n}.(T_{m.}) = T_{n.m}.$$

$$T_{n.}.U_{m.} = \frac{1}{2}.[U_{n..+m.} - U_{n.-m.}]$$

$$T_{n.}.V_{m.} = \frac{1}{2}.[V_{n.+m.} + T_{n.-m.-1}]$$

$$T_{n.}.W._{m.} = \frac{1}{2}.[W_{n.+m.} + W_{n.-m.-1}]$$

$$\mathrm{(x+1).V_{m.}.V_{n.} = T_{m.+n.+1} + T_{m.-n.}}$$

$$\mathrm{(x-1).W_{m.}.W_{n.} = T_{m.+n.+1} - T_{m.-n.}}$$

$$\mathrm{V_m.W_{n.} = U_{m.+n.+1} - U_{m.-n}.}$$

Additional useful form

$$\mathrm{2T_{m.}.V_{n.} = V_{m.+n.} + V_{m.-n.-1} = V_{m.+n.} + V_{n.-m.}}$$

$$\mathrm{2T_m..W_{n.} = W_{m.+n.} - W_{m.-n.-1} = W_{m.+n.} + W_{n.-m.}}$$

$$\mathrm{2(x+1).U_m.V_{n.} = W_{m.+n.} + W_{m.-n.-1}}$$
$$\mathrm{= W_{m.+n.} - W_{n.-m}.}$$

$$\mathrm{2(x-1).U_m..W_{n.} = V_{m.+n.} - V_{m.-n.-1}}$$
$$\mathrm{= V_{m.+n.} - V_{n.-m}.}$$

$$V_{-n.}(x) = V_{n.-1}(x), \qquad W_{-n.}(x) = -W_{n.-1}(x)$$

The derivatives also exhibit elegant relationships:

$$\frac{d}{dx}T_{n.} = n.U_{n.-1}$$

$$\tfrac{d}{dx}T_{n.} = \tfrac{n.}{(1-x^2)}T_{n.-1} - \tfrac{n.x}{(1-x^2)}T_{n.}$$

$$\tfrac{d}{dx}U_{n.} = \tfrac{n.+1}{(1-x^2)}.U_{n.-1} - \tfrac{n.x}{(1-x^2)}U_{n.}$$

$$\tfrac{d}{dx}\left[\sqrt{1-x^2}\,.U_{n.-1}\right] = -\boldsymbol{n.}\tfrac{1}{\sqrt{1-x^2}}T_n$$

$$\frac{d}{dx}\left[\sqrt{1+x}\,.V_{n.}\right] = \left(n.+\frac{1}{2}\right)\frac{1}{.\sqrt{1+x}}W_{n.}$$

$$\tfrac{d}{dx}\left[\sqrt{1+x}.\ W_{n.}\right] = -\left(n.+\tfrac{1}{2}\right).\tfrac{1}{\sqrt{1-x}}.V_{n.}$$

These obtained by differentiating the trigonometric. For instance

$$\frac{d}{dx}T_{n.}(x) = \frac{d}{dx}(\cos.\mathrm{n.}\theta) = -\frac{1}{\sin\theta}(-n.\sin.\theta)$$
$$= n.U_{n.-1}(x)$$

Which

$$= \tfrac{1}{(1-x^2)}((1-x^2)U_{n.-1})$$
$$= \tfrac{1}{(1-x^2)}(n.xT_{n.} - nT_{n.+1})$$
$$= \tfrac{1}{(1-x^2)}(n.xT_{n} - 2xnT_{n.} + n.T_{n.-1})$$
$$= \tfrac{n}{(1-x^2)}T_{n.-1} - \tfrac{n.x}{(1-x^2)}T_{n.}$$

Some special values that is important

$$T_{n.}(1) = 1,\ \ T_{n.}(-1) = (-1)^{n.},\quad T_{n}.(-x).$$
$$= (-1)^{n.}.T_{n.}(x)$$

$$T_{2n.}(0) = (-1)^{n}.,\ \ T_{2n.+1}(0) = 0$$

$$\frac{d}{dx}T_{n.}(-1) = (-1)^{n.-1}n.^2, \qquad \frac{d}{dx}T_{n.}(1) = n.^2$$

$$\frac{d^2}{dx^2}T_{n.}(1) = \frac{n.^2.(n.^2-1)}{3.} = (-1)^{n.}\frac{d^2}{dx^2}T_{n.}(-1)$$

$$U_{n.-1}(1) = n.,\ \ U_{n-1}(-1) = -n.(-1)^{n.}$$

$$\frac{d}{dx}U_{n.-1}(x_j) = \frac{n.(-1)^{j.+1}}{1-x_{j.}^2},$$

$$\frac{d}{dx}U_{n.-1}(1) = \frac{n.(n^2-1)}{3}$$

$$V_{n.}(1) = 1, \qquad V_{n.}(-1) = (-1)^{n.}.(2n.+1),$$

$$V_{n.}(-x) = (-1)^{n.}.V_{n.}(x)$$

$$V_{2n.}(0) = (-1)^{n.},\ \ V_{2n.+1}(0) = (-1)^{n}.$$

$$W_{n.}(1) = \mathrm{2n.}+1, \qquad W_{n.}(-1) = (-1)^{n.},$$

$$W_{n.}(-x) = (-1)^{n}.W_{n.}(x)$$

$$W_{2n.}(0) = (-1)^{n.}, \qquad W_{2n.+1}(0) = (-1)^{n.}$$

$$W_{n.}(1) = (-1)^{\boldsymbol{n.}}\ V_{n.}(-1) = 2n.+1,\ \ V_{n.}(1)$$
$$= (-1)^{n.}.W_{n.}(-1) = 1$$

$$d^kT_{n..}(1) = \prod_{m.=0}^{k.-1}\frac{(n.-m.).(n.+m.)}{2m.+1}$$
$$= \frac{n..(n.+k-1)!.\sqrt{\pi}}{(n.-k)!.2^{k.}.\Gamma(k+\frac{1}{2})}.$$
$$d^kU_{n.}(1) = (n.+m.)\prod_{m.=0}^{k.-1}\frac{(n.-m.).(n.+m.+2)}{2m.+3}$$
$$= \frac{n..(n.+k-1)!.\sqrt{\pi}}{(n.-k)!.2^{k+1.}\Gamma(k+\frac{3}{2})}d^kV_{n.}(1)$$
$$= (-1)^{n.+k}d^k.W_{n.}(-1)$$
$$= \prod_{m=0}^{k.-1}\frac{(n.-m.).(n.+m.+1)}{2m.+1},$$
$$d^kW_{n.}(1) = (-1)^{n.+k.}d^kV_n(-1)$$
$$= (2n + 1).\prod_{m.=0}^{k.-1}\frac{(n.-m.)(n.+m.+1)}{2m.+3},$$
$$V_{n.}(1) = 1,\quad V_n.(-1) = (-1)^{n.}(2n.+1),\quad W_{n.}(1) = (2n+1),$$
$$,\quad W_n(-1) = (-1)^{n.},$$
$$\frac{d}{dx}^k V_{n.}(1) = \frac{\sqrt{\pi}2^{-k}.(n.+k)!}{(n-k)!.}\frac{1}{\Gamma(k+\frac{1}{2})}$$
$$\frac{d}{dx}^k V_{n.}(-1) = \frac{\sqrt{\pi}2^{-k}(n+k)}{(n-k)}\frac{(-1)^{n+k}.(1+2n)}{2\Gamma(k+\frac{3}{2})}$$
$$\frac{d}{dx}^k W_{n.}(1) = \frac{\sqrt{\pi}2^{-k}(n+k)}{(n-k.)}\frac{(1+2n.)}{2\Gamma(k+\frac{3}{2})}$$
$$\frac{d}{dx}^k W_{n.}(-1) = \frac{\sqrt{\pi}.2^{-k.}(n+k)!}{(n.-k)}.\frac{(-1)^{n.+k}}{\Gamma(k+\frac{1}{2})}$$

Some estimation

$$|T_n.| \le 1,\quad |T_{n.+1} - T_{n.-1}| \le 2$$
$$|W_{n.}| \le 2n.+1,\quad |V_{n.}| \le 2n.+1$$
$$\left|\frac{d}{dx}T_{n.}\right| \le n^2,\quad \left|\frac{d}{dx}T_{n.}(\pm 1)\right| = n^2,\quad \left|\frac{d^2}{dx^2}(T_{n.})\right| \le \frac{n^{2.}.(n^2-1)}{3}$$
$$\left|T_{n.}^{(k)}(x)\right| \le \prod_{k=0}^{k-1}\frac{n^2-r^2}{(2r+1)!},.\qquad T_{n.}^{(k)}(x) \le T_{n.}^{(k)}(1)$$

The identities presented above form a coherent and way. Their proof is not difficult, it is straightforward trigonometric manipulation.

5. The explicit Representations

Each kind can be expressed explicitly through

$$T_{n.}(x) = \frac{n}{2}\sum_{s=0}^{[n./2]}(-1)^s.\frac{(n.-s-1)!.}{s!(n-2s)!}.(2x)^{n-2s}$$
$$U_{n.}(x) = \sum_{s=0}^{[n./2]}(-1)^s..\frac{(n-s)!}{s!(n-2s)!}.(2x)^{n-2s} \qquad (16)$$
$$V_{n.}(x) = \frac{n.}{2}\sum_{s=0}^{\left[\frac{n.}{2}\right]}(-1)^s.\frac{(n.-s)!.}{s!(n.-2s)!.}.(2x)^{n.-2s}$$
$$W_{n.}(x) = \frac{n.}{2}\sum_{s=0}^{\left[\frac{n.}{2}\right]}(-1)^s.\frac{(n.-s)!.}{s!(n.-2s)!.}.(2x)^{n.-2s}$$

For the third and fourth kind, we utilize the identities:

$V_n = U_n - U_{n-1}$, $W_n = U_n + U_{n-1}$
Then, we have

$$V_{n.}(x) = \sum_{s=0}^{[n./2]}(-1)^s\frac{(n.-s)!.}{s!(n.-2s)!.}(2x)^{n.-2s} - \sum_{s=0}^{[n./2]}(-1)^s\frac{(n.-s-1)!}{s!(n.-1-2s)!}.(2x)^{n.-1-2s}$$
$$W._n(x) = \sum_{s=0}^{\left[\frac{n}{2}\right]}(-1)^s.\frac{(n.-s)!.}{s!(n-2s)!}.(2x)^{n.-2s} + \sum_{s=0}^{[n/2]}(-1)^s\frac{(n.-s-1)!}{s!.(n.-1-2s)!}..(2x)^{n.-1-2s}$$

For evenn and odd indices, more compact forms exist:
Theorem1.5: (Kishora &Verma, 2023) For integer n ≥ 0, the explicit series representations:

1. $T_{2n.}(x) = \sum_{k=0}^{n.}\frac{(-1)^{n.-k}.2^{2k}n}{(k+n)}.\binom{n.+k}{2k}.(x)^{2k}$
2. $T_{2n.+1} = \sum_{k=0}^{n}\frac{(-1)^{n.-k..}.2^{2k}.(2n.+1)}{(k+n.+1)}.\binom{n.+k+1}{2k+1}.(x)^{2k+1}$
3. $U_{2n.}(x) = \sum_{k=0}^{n.}\frac{(-1)^{n.-k}.2^{2k}.(2k+1)}{(k+n.+1)}.\binom{n.+k+1}{2k+1}.(x)^{2k}$
4. $U_{2n.+1}(x) = \sum_{k=0}^{n.}\frac{(-1)^{n.-k}2^{2+k}(k+1)}{(k+n.+2)}\binom{n.+k+2}{2k+2}(x)^{2k+1}$
5. $U_{2n.-1}(x) = \sum_{k=0}^{n.-1}\frac{(-1)^{n.-k-1}2^{2k+2.}(k+1).}{(k+n.+2)}.\binom{n.+k+1}{2k+2}.(x)^{2k+1}$
6. $V_{2n.}(x) = (2x)^{2n.} + \sum_{k=0}^{n.-1}(-1)^{n.-k}.2^{k+1}.\binom{n.+k}{2k}.\left[\frac{(x)^{2k}}{2} + \frac{(n.-k)}{(2k+1)}.(x)^{2k+1}\right]$
7. $V_{2n.+1}(x) = \sum_{k=0}^{n.}(-1)^{n.-k}2^{2k+1}(2k+1)\binom{n.+k}{2k}\left[\frac{n.+k+1}{2k+1}(x)^{2k+1} - \frac{(x)^{2k}}{2}\right]$

8. $W_{2n.}(x) = (2x)^{2n.} + \sum_{k=0}^{n.-1}(-1)^{n.-k}2^{2k+1}\binom{n.+k}{2k}\left[\frac{(x)^{2k}}{2}.(x)^{2k+1} - \frac{(n.-k)}{(2k+1)}\right]$
9. $W_{2n.+1}(x) = \sum_{k=0}^{n.}(-1)^{n.-k}.2^{2k+1}.\binom{n.+k}{2k}.\left[\frac{(n.+k+1)}{(2k+1)}.(x)^{2k+1} + \frac{(x)^{2k}}{2}\right]$

(17)

We derive these formulas using the known relations between the four kinds and their trigonometric definitions.

Proof: - For T_{2n} and T_{2n+1} , we start from trigonometric definition, expanding $\cos(n\theta)$ via De Moivre and using the binomial theorem gives the explicit sums. After simplification, the expressions above are obtained. (Alternatively, one can use the recurrence and known explicit forms)

For $U_{2n.}$ and $U_{2n.+1}$, using

$\frac{d}{dx}T_{n.} = n.U_{n.-1}$, $\frac{d}{dx}T_{2n.} = 2n.U_{2n.-1}$

Differentiating the series for $T_{2n.}$ term wise with shiftting the index back. The formula for $U_{2n.+1}$ follows similarly from $T_{2n.+1}$

$$T'_{2n.+1} = \sum_{k.=0}^{n.}(2k + 1)\frac{(-1)^{n.-k}.2^{2k.}.(2n.+1)}{(k+n.+1)}.\binom{n.+k+1}{2k+1}.(x)^{2k+1}$$

Using $\frac{d}{dx}T_{2n.+1} = (2n.+1).U_{2n.-1}$

For $V_{2n.}$ and $V_{2n.+1}$

$$V_{n.}(x) = \frac{cos(n.+\frac{1}{2})\theta}{cos\frac{\theta}{2}}, \qquad V_{n.} = U_{n.} - U_{n.-1},$$

$$V_{2n.} = U_{2n.} - U_{2n.-1}$$

$$V_{2n.}(x) = \sum_{k.=0}^{n.}\frac{(-1)^{n.-k}.2^{2k}.(2k+1)}{(k.+n.+1)}\binom{n.+k.+1}{2k+1}.(x)^{2k.} - \sum_{k=0}^{n.-1}\frac{(-1)^{n.-k.-1}.2^{2k.+2}.(k.+1)}{(k.+n.+1)}\binom{n.+k.+1}{2k.+2}.(x)^{2k.+1}$$

$$= \sum_{k=0}^{n}(-1)^{n-k}2^{2k}\binom{n+k}{2k}(x)^{2k} - \sum_{k=0}^{n-1}\frac{(-1)^{n.-k-1}.2^{2k.+1}.(n.-k)}{(2k.+1)}.\binom{n.+k}{2k.}.(x)^{2k+1}$$

$$= (2x)^{2n.} + \sum_{k=0}^{n-1}(-1)^{n.-k}.2^{2k.}\binom{n.+k}{2k}(x)^{2k} + \sum_{k.=0}^{n.\_1}(-1)^{n.-k}.2^{2k+1}.\binom{n.+k}{2k}.(x)^{2k+1}$$

$$= (2x)^{2n.} + \sum_{k=0}^{n.-1}(-1)^{n.-k}.2^{2k+1}\binom{n.+k}{2k}\left[\frac{(2x)^{2k}}{2} + \frac{(n.-k)}{(2k+1)}.(x)^{(2k+1)}\right]$$

The odd case $V_{2n.+1} = U_{2n.+1} - U_{2n.}$ is treated analogously.

$$V_{2n.+1}(x) = \sum_{k.=0}^{n.}\frac{(-1)^{n.-k}.2^{2k+2.}.(k+1).}{(k+n+2)}\binom{n.+k+2}{2k+2}.(x)^{2k+1} - \sum_{k.=0}^{n.}\frac{(-1)^{n.-k.}.2^{2k}.(2k+1).}{(k+n+1)}\binom{n.+k+1}{2k+1}.(x)^{2k}$$

$$= \sum_{k.=0}^{n.}\frac{(-1)^{n.-k}.2^{2k+1}.(n.+k+1).}{(2k.+1)}\binom{n+k}{2k}.(x)^{2k+1} - \sum_{k=0}^{n}(-1)^{n-k}2^{2k}\binom{n+k}{2k}(x)^{2k}$$

$$= \sum_{k.=0}^{n.}(-1)^{n.-k}.2^{2k+1}\binom{n.+k}{2k}.\left[\frac{n.+k+1}{2k+1}.(x)^{2k+1} - \frac{(x)^{2k}}{2}\right]$$

For $W_{2n.}$ and $W_{2n.+1}$, from the symmetry relation

$$W_{n.}(x) = (-1)^{n.}.V_{n.}(-x)$$

$$W_{2n.}(x) = (-1)^{2n.}.V_{2n.}(-x) = V_{2n.}(-x)$$

Changing x by $-x$ for $V_{2n.}(x)$

$$V_{2n.}(x) = (2x)^{2n} + \sum_{k=0}^{n.-1}(-1)^{n.-k..}2^{2k+1}\binom{n.+k}{2k}.\left[\frac{(-x)^{2k}}{2} + \frac{(n.-k)}{(2k+1)}.(-x)^{2k+1}\right]$$

$$= (-2x)^{2n.} + \sum_{k.=0}^{n.-1}(-1)^{n.-k}.2^{2k+1}\binom{n.+k}{2k}\left[\frac{(x)^{2k}}{2} - \frac{(n.-k)}{(2k+1)}.(x)^{2k+1}\right]$$

$$= (2x)^{2n.} + \sum_{k.=0}^{n.-1} (-1)^{n.-k} 2^{2k+1} \binom{n.+k}{2k} \left[\frac{(x)^{2k}}{2} - \frac{(n.-k)}{(2k+1)} . (x)^{2k+1} \right]$$

Gives formula for $W_{2n.}(\boldsymbol{x})$

$$W_{2n.}(\boldsymbol{x}) = (2x)^{2n.} + \sum_{k=0}^{n.-1} (-1)^{n.-k} . 2^{k+1} \binom{n.+k}{2k} . \left[\frac{(x)^{2k}}{2} - \frac{(n.-k)}{(2k+1)} . (x)^{2k+1} \right]$$

The odd case follows similarly

$$W_{2n.+1}(x) = (-1)^{2n.+1} . V_{2n+1}(-x) = V_{2n.+1}(-x)$$

These explicit formulas are particularly useful for numerical evaluation and for deriving further identities, such as the derivative expansions given in the next section.

Proposition.1. 5: (Verma et al., 2024) For the r-th derivative

$$T^{(r)}{}_{2n.}(x) = \sum_{k.=\left[\frac{r}{2}\right]}^{n.} \frac{(-1)^{n.-k.} . 2^{2k.} . n.}{(n.+k.)} \binom{n.+k.}{2k.} (x)^{2k.-r}$$

$$U^{(r)}{}_{2n.}(x) = \sum_{k.=\left[\frac{r}{2}\right]}^{n.} \frac{(-1)^{n.-k.} . 2^{2k} . (n.+k.)!}{(n.-k.)!. (2k.-r.)!} \binom{n.+k.+1}{2k.+1} (x)^{2k.-r}$$

$$V^{(r)}{}_{2n\cdot}(x) = \sum_{k.=\left[\frac{r}{2}\right]}^{n.} \frac{(-1)^{n.-k} . 2^{2k.} (n.+k.)!}{(n.-k.)!. (2k-r.)!} (x)^{2k.-r} + \sum_{k.=\left[\frac{r.-1}{2}\right]}^{n.-1} \frac{(-1)^{n.-k.} 2^{2k.+1} . (n.+k.)!}{(n..-k-1)!. (2k-r+1)!} (x)^{2k.-r}$$

$$W^{(r)}{}_{2n.}(x) = \sum_{k.=\left[\frac{r}{2}\right]}^{n.} \frac{(-1)^{n.-k.} 2^{2k.} (n.+k.)!.}{(n.-k.)!. (2k.-r.)!} . (x)^{2k.-r} + \sum_{k.=\left[\frac{r-1}{2}\right]}^{n.-1} \frac{(-1)^{n.-k.} 2^{2k.+1} (n.+k.)!}{(n.-k.-1)! (2k.-r+1)!} (x)^{2k.-r}$$

Proof: The formulas above are obtained by differentiating the explicit series representations and simplifying the resulting binomial coefficients.

6. The Monomial Expansion and Related Formula

In spectral methods and approximation theory, one frequently needs to represent a monomial x^n as a sum of Chebyshev polynomials of various kinds. Such expansions allow the conversion between the standard power basis and the orthogonal Chebyshev bases, which is crucial for constructing differentiation matrices, solving differential equations, and implementing efficient quadrature rules. See also (Mason & Handscomb, 2003).

Theorem1.6. The $x^n.$ can be expanded as

$$x^{n.} = \sum_{k=0}^{n.} a_k . T_{k.} = \frac{n!.}{2^{n.-1}} \sum_{k=1}^{n.} \frac{T_{k.}}{\left(\frac{n.+k.}{2}\right)!.(\frac{n.-k.}{2})!}$$

Proof: By the orthogonality and the Rodrigues formula

$$a_{k.} = \frac{c_{k.}.}{\pi} \frac{(-1)^{k.} . 2^{k.} . k!.}{(2k)} . \int_{-1}^{1} (1 - x^2)^{1/2..} \left(\frac{d^k}{dx^k} (1 - x^2)^{k.-1/2} \right) x^{n.} . dx$$

$$= \frac{c_k}{\pi} \frac{.(-1)^k.2^{k.}k!.}{(2k)} (-1)^k \frac{n!}{(n-k)!} \int_0^1 (1-x^2)^{k-1/2} x^{n-k} dx$$

$$= \frac{c_k}{\pi} \frac{(-1)^k 2^k k!.}{(2k)} . (-1)^k \frac{n!.}{(n-k)} \frac{1+(-1)^{n.-k}}{2} \int_0^1 (1-t)^{\frac{k+1}{2}-1} t^{\frac{n-k+1}{2}-1} dt$$

$$= \frac{c_{k.}}{\pi} \frac{(-1)^{k..} 2^k .k!.}{(2k)!.} (-1)^k \frac{n!.}{(n-k)!.} \frac{(1+(-1)^{n.-k}) \Gamma(\frac{k+1}{2}) \Gamma(\frac{n.-k+1}{2})}{2\Gamma(\frac{n-k+1}{2})}$$

$$= \frac{c_{k.}}{\pi} \frac{(-1)^k .2^k .k!}{(2n)!.} . (-1)^{k.} \frac{n.!.}{(n.-k)!.} \frac{(1+(-1)^{n.-k})}{2} \frac{(n-k)!.(2k).!.\pi}{2^{n.+k}.\left(\frac{n.+k}{2}\right)!.\left(\frac{n-k}{2}\right)!.k!}$$

After integrating by parts, the integral reduces to a Beta function

$$B\left(\sqrt{\frac{k+1}{2}}, \sqrt{\frac{n-k+1}{2}}\right) = \frac{\sqrt{\frac{k+1}{2}} \sqrt{\frac{n-k+1}{2}}}{\sqrt{\frac{k+1}{2} + \frac{n-k+1}{2}}} = \frac{\sqrt{\frac{k+1}{2}} \sqrt{\frac{n-k+1}{2}}}{\sqrt{\frac{n+2}{2}}}$$

If $n-k$ is odd, $a_k = 0$, $n-k$ is even, simplification yields

$$a_k = \frac{n! c_k}{2^n (\frac{n+k}{2})! (\frac{n-k}{2})!}$$

$$c_{0.} = 1, \ (c_{1.} = 2, \qquad k \geq 1)$$

Theorem.2.6. For $x^{n\cdot}$ expressed in the second kind basis

$$x^{n\cdot} = \sum_{k=0}^{n} a_k . U_{k.} = \frac{n!.}{2^{n.}} \sum_{k=1}^{n.} \frac{(k.+1) U_k}{\left(\frac{n.+k+2}{2}\right)!.\frac{n.-k}{2})!}$$

Proof.: The same way from orthogonality and the Rodrigues formula, therefore

$$b_{k.} = \frac{c_k}{\pi} . \frac{(-1)^k 2^{k+1}(k+1)!}{(2k+1)!.} . \int_{-1}^{1} \left(\frac{d^k}{dx^k} (1-x^2)^{\frac{k+1}{2}} \right) x^{n.} dx$$

$$= \frac{c_k}{\pi} \frac{(-1)^k 2^{k+1.}(k+1).!.}{(2k+1)!} . (-1)^{k.} \frac{n!.}{(n.-k)!} \int_0^1 (1 - x^2)^{\frac{k+1}{2}} x^{n.-k} dx$$

$$= \frac{c_k}{\pi} \frac{(-1)^k . 2^{2k.+1} . (k.+1)!}{(2k+1)!} (-1)^{k.} . \frac{n!.}{(n.-k)!.} . \frac{(1+(-1)^{n.-k}) . \Gamma\left(\frac{k+1}{2}\right) . \Gamma\left(\frac{n.-k+1}{2}\right)}{2 . \Gamma\left(\frac{n-k+1}{2}\right)!.}$$

$$= \frac{c_{k.}}{\pi .} \frac{(-1)^k . 2^k . k!}{(2n.)!.} (-1)^k . \frac{n!.}{(n.-k)!} . \frac{(1+(-1)^{n.-k})}{2} \frac{(n.-k)!.(2k).!.\pi}{2^{n.+k} . \left(\frac{n.+k}{2}\right).!.\left(\frac{n.-k}{2}\right)!k!}$$

After integrating by parts and evaluation of the resulting Beta integral, one obtains

$$b_{k.} = \frac{(n.-k)!\,(2k+2)!\,\pi}{2^{n.+k+2} \left(\frac{n.+k+2}{2}\right)!\,\left(\frac{n-k}{2}\right)!\,(k+1)!}$$

For $n.-k$ even, otherwise $b_k = 0$.

The monomial x^n also may expressed as a finite series in either the third-kind basis $V_k(x)$or the fourth-kind basis $W_k(x)$.

Theorem.3.6. The monomial $x^{n.}$. is expanded as a finite series in the third kind $V_{k.}(x)$

$$x^{n.} = \sum_{k.=0}^{\infty} c_{k..} . V_{k.} = 2^{-n.} \sum_{m=0}^{[n./2]} \binom{n.}{m.} V_{n.-2m.} + 2^{-n.} \sum_{m=0}^{[(n.-1)/2]} \binom{n.}{m.} V_{n.-1-2m.}$$

Proof: Starting from the Rodrigues formula

$$c_k = \frac{2}{\pi} \frac{(-2)^{k.} . (k).!.}{(2k)!.} . \int_{-1}^{1} x^{n.} . \frac{d^k}{dx^k} \left((x^2 - 1)^k . \sqrt{\frac{1+x}{1-x}} \right) . dx$$

Applying integration by parts repeatedly k times, the boundary terms vanish at $x = \pm 1$, due to the factor $(1-x^2)^{k.}$. This yield

$$c_k = \frac{2}{\pi} \frac{(-2)^k . (k).}{(2k)!.} . (-1)^{k.} \frac{((n.).!.}{(n.-k)!.} \int_{-1}^{1} x^{n.-k.} (1 - x^2)^k . \sqrt{\frac{1+x}{1-x}} . dx$$

Since $\frac{d^k}{dx^k}(x^{n.}) = \frac{(n.)!.}{(n.-k)!.} . x^{n.-k}$ for $n \geq k$, this become Now observe that

$$(x^2-1)^{k.} \sqrt{\frac{1+x}{1-x}} = (-1)^k . (1-x^2)^{k.} . \sqrt{\frac{1+x}{1-x}} = = (-1)^{k.} . (1-x^2)^{k.-\frac{1}{2}} . (1+x)$$

Then

$$c_{k.} = \frac{2}{\pi} \frac{(2)^{k.} . (k.)!\,(n)!}{(2k)! . (n.-k.).!.} \int_{-1}^{1} x^{n.-k.} (1+x) . (1 - x^2)^{k.-1/2} dx$$

This integral may evaluate as a Beta integral. After simplification, one obtains the required one.

Theorem.4.6: The monomial x^n is expanded as a finite series in the fourth kind $W_k(x)$

$$x^{n.} = \sum_{k=0}^{n.} d_k . W_{k.} = 2^{-n.} \sum_{m=0}^{[n./2]} \binom{n.}{m.} W_{n.-2m.} - 2^{-n.} \sum_{m=0}^{[(n-1)/2]} \binom{n.}{m.} W_{n.-1-2m.}$$

Proof: By the Rodrigues formula, and following a similar way, we obtain

$$d_k = \frac{c_k}{\pi} \int_{-1}^{1} x^n W_k . \sqrt{. \frac{1-x}{1+x}} . dx$$

$$= \frac{(-2)^{k.} . (k).!.}{(2k)!.} \int_{-1}^{1} x^{n.} \frac{d^k}{.dx^k} \left(. (x^2 - 1)^{k.} . \sqrt{\frac{1-x}{1+x}} \right) . dx$$

$$= \frac{(2)^{k.} . (k)!. (n.)!,}{(2k.)! . (n.-k)!,} . \int_{-1}^{1} x^{n-k} (1-x)\,(1-x^2)^{k-1/2} dx$$

Theorem.5.6. For all positive integer $n, q \geq 0$

$x^q V_{n.} = \frac{1}{2^q} \sum_{t=0}^{q} \binom{q}{t} V_{n+q-2t}$

$$x^{q+1} V_n = \frac{1}{2^{q+1}} \sum_{t=0}^{m+1} \binom{q+1}{t} V_{n+q-2t+1}$$

Proof: By the induction hypothesis on m. The case $q = 0$ is trivial. Assume it holds for a given m and using the recurrence $x.V_{k.} = \frac{1}{2}[V_{k.+1} + V_{k.-1}]$, we get

$$x\left(\frac{1}{2^{.q}}\sum_{t=0}^{q.}\binom{q.}{t}V_{n+q.-2t}\right) = \frac{1}{2^{q+1}}\sum_{t=0}^{q.}\binom{q.}{t}.\,[V_{n+q-2t+1} + V_{n+q-2t-1}]$$

Separate the sums and shift the index in the second sum

$$x^{q.+1}V_{n.} = \frac{1}{2^{q+1}}\left[\sum_{t=0}^{q.}\binom{q.}{t}V_{k+q.-2t+1} + \sum_{t=1}^{q+1}\binom{q}{t-1}V_{n+q-2t+1}\right]$$

$$= \frac{1}{2^{q.+1}}\left[V_{n+q+1} + V_{n-q-1} + \sum_{t=1}^{q.}\binom{q.+1}{t}V_{n+q.-2t+1}\right]$$

Analogous formulas hold for W_n

$$x^qW_{n.} = \frac{1}{2^q}\sum_{s=0}^{q.}\binom{q.}{s}W_{n+q-2s}$$

The proof is identical to that for V_n using $W_{n.}(x) = (-1)^{\boldsymbol{n.}}.V_{n.}(-x)$ and

$$x.W_{k.} = \frac{1}{2}[W_{k.+1} + W_{k.-1}]$$

Further important properties involve expressing monomials x^m in the Chebyshev basis (inversion) and finding the coefficients for products of polynomials (linearization). Substituting $n = 0$ in the relations yields

$$x^q = \frac{1}{2^q}\sum_{t=0}^{q.}\binom{q}{t}V_{q-2t}$$

7. Conclusion

Chebyshev polynomials are beautiful because of its ability generate new formulas and identities. Their recurrence relations, orthogonality and connection to trigonometry make them a valuable insight in mathematics. We have presented a unified Chebyshev polynomials treatment of the first four kinds, emphasizing their interconnections through generating functions, Rodrigues formulas, derivative expansions, and monomials inversions. Some explicit series for even and odd indices, as well as derivative expansions extend the existing ones.

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