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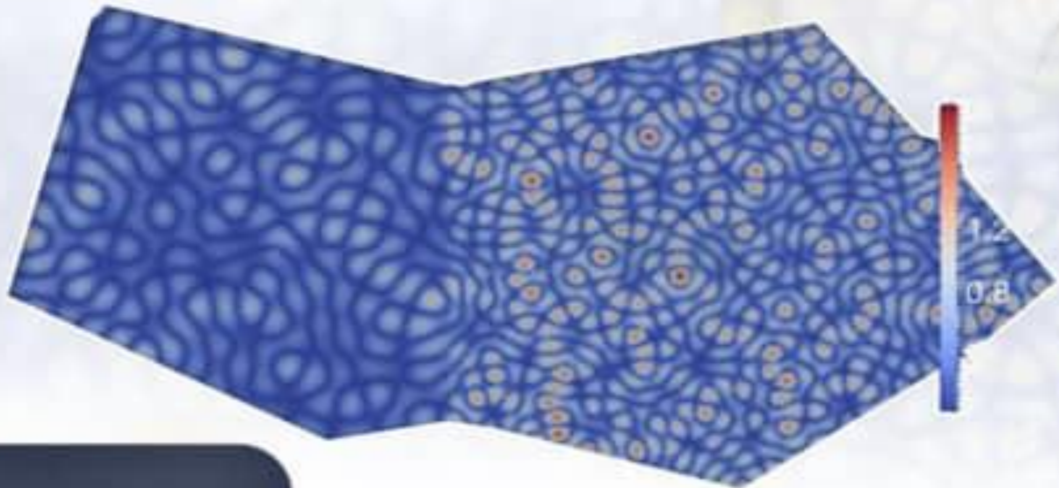

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Generating Matrices of Rotations in Minkowski Spaces using the Lie Derivative

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A B S T R A C T

This paper aims to generate materices of rotations in Minkowski using the Lie Derivative. The calculus on manifolds in Lorentzian spaces are used to generate matrices of rotation in three-dimensional Lorentz-Minkowski space which includes one axis in timelike and the other two are spacelike axes. The findings showed that the manifolds and their calculus dramatically increased the use of Lie derivative in many branches of mathematics and physics, The findings also revealed that matrices (of rotation) leave one line (axis) fixed and these matrices of rotation are used widely in differential geometry in physics. Furthermore, the findings demonstrated that any surfaces of revolution inside this space must be invariant under one of these matrices. The main result of this paper is a new procedure of creating rotational matrices explicitly using the Lie derivative and deriving it into a linear system of ordinary differential equaion. Solving this system leads to matrices of rotation that leaves one axis fixed in Minkowski space.

1 Introduction

Manifolds and their applications combine many critical areas of mathematics; they generalize concepts such as curves, surfaces, and ideas from linear algebra and topology.

In differential geometry, Lie (1842, 1899) evaluates the change of a tensor field (including scalar functions, vector fields, and one-forms) along the flow defined by another vector field. This change is invariant coordination. Therefore, the Lie derivative is defined by any differentiable manifolds. The Lie derivative is the early beginning of calculus on manifolds.

On the other hand, the rotational surfaces in Euclidean space have been studied for a long time. Many examples of curves on these surfaces have been discovered. These surfaces are called rotating a plane curve on an axis (called the axis of rotation). This rotation is done using a matrix of rotation, which is called a one-parameter group of isometry and orthogonal matrix SO(3) (For more information, see Pressley's differential geometry textbook written by Pressly, 2010).

However, in the last three decades, the vision of geometry has been developed to other spaces including the time axis. While in Euclidean space all axis are spacelike vectors (+,+,+), in (Minkowski) space there is a one-time axis (+,+,-). This different sign changes all theorems of inner and outer products of vectors to another field (See Hall, 2004; Hilgert and Neeb, 2011; Lopez, 2014; Saad, 2016).

Minkowski's space is more complicated compared to Euclidean's. We can distinguish other types of matrices of rotation which are later called space,time, and null rotation.

Thus, there are three types of one-parameter groups of isometries. By considering the rigid motion of ambient space that keeps a line fixed, we will solve these types of rotation to generate these different types explicitly. Firstly, similar to each axis of rotation, there is a matrix of rotation that leaves a (type) of axis fixed.

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Furthermore, we use the definition of Lie derivative and *Killing* (1847-1923). Killing fields are the infinitesimal generator of isometries; that is, flows generated by Killing fields are continuous isometries of the manifold and matrices of rotation generate in three-dimensional Minkowskian spaces.

This work is located between differential geometry and physics. This paper also tries to show explicitly the relationship between differential geometry and Lorentz groups, i.e. Lorentzian Manifolds.

Rotation in E^3 preserves all distance. As a consequence, they preserve all inner products. Also, the map transfers the orthonormal basis to another orthonormal basis with a linear transformation of finite dimensional vector space. Rotation can be represented by a matrix once a basis is chosen.

In 3D Euclidean space, restricting the attention to the proper rotation, we find that the set of all 3×3 matrices satisfies the orthogonal-unit matrix which is known by $SO(3)$.

The group of $SO(3)$ is a group with an identity element of unit matrix I and the matrix multiplication as the group operation, then $SO(3)$ defines a rotation group of E^3.

Any matrix $R \in SO(3)$ has been taken into account an eigenvector with an eigenvalue of 1. It gives the axis of rotation.

In this paper, as previously mentioned, we go forward to another space which is the 3D Minkowski space. While the three dimensional in Euclidean space are all positive definite in the self-inner product, in Minkowski space there is a one-time axis giving the possibility of minus singe of self-inner product. Therefore, not all axis of rotation is the same, and then we go through all possibility of the axis of rotation upon the Minkowski space criteria.

Accordingly, using the Lie derivative of diffeomorephisom functions gives a system of ODE. Therefore, three different types of matrices simulate the matrix $R \in SO(3)$, but now for Minkowski spaces i.e. $R \in SO(2,1)$. (two spacelike and one timelike vectors). Matrices of rotations generate in Minkowski space which is the beginning of the geometry of surfaces in Minkowski space.

In sections two and three, we define the isometry in Lorentz groups, rotation, and the transformation of Lorentz groups.

Section four includes the main definition of calculus on the manifold which describes the Lie derivative, its properties, and the generators of rotation in Lorentz space seeking to Killing vector fields.

Finally, section five, presenting the main results of this paper, covers in detail all the computation parts of generating rotational matrices in three-dimensional Minkowskian spaces using systems of ordinary differential equations.

2 The Isometry

In this section, we review the isometry in E^3 and consider that for E_1^3 (three dimensional Minkowskian space). An isometry is defined as a function that preserves the matric (Carrol,1997; Torres, 2012).

2.1 Introduction

Definition: A diffeomorphism between two Riemannian manifold $\varphi: (M, g) \to (N, h)$ is an isometry if $\varphi^* h = g$ which means for every point $x, d_x\varphi$.

Linear map isometry is between $T_x M$ and $T_{\varphi(x)} N$ that leaves a line (length) fixed .

A particular consequence of this definition is that:

Definition: $f: (M, g) \to (M, g)$ is a diffeomorphism from a manifold onto itself, with the property that is for all $p \in M$ and all $V, W \in T_p M$

$$g_{f(p)}\left(f_p^* V, f_p^* W\right) = g_p(V, W) \tag{1}$$

Then f is an isometry of (M, g).

2.2 Isometry Group

The isometry group of the Riemannian manifold (M, g) is the set of diffeomorphism of M that is g-isometric, e.g. the Lorentz group.

2.3 Rotation

Rotation in three dimensional Euclidean space E^3 preserves all distance .i.e. they are isometries. A rotation can be represented by a matrix . All rotational matrices are special orthogonal matrices ,which in $\det R = 1$ and $R^T = R^{-1}$.

In this section we define the rotation as well as mention standard rotations in three dimensional Euclidean space E^3 (For more detail, see Hall,2004; Hilgert & Neeb, 201; Lopez, 2014).

3 The Lorentz groups and Transformation

In this section, we define the Lorentz group, Lorentz transformation and types of Lorentz transformation in three dimensional Lorentz Minkowski space E_1^3 .

- The Lorentz Group

Lorentz group is the group of isometries of Lorentz Minkowski space which preserve the origin.

- The Lorentz Transformation

The Lorentz transformation for any position (point) in Lorentz Minkowski space is defined by

$$\widetilde{S} = \Lambda S \quad \text{where } g = \Lambda^T g\, \Lambda \tag{2}$$

Where Λ^T is the transposed matrix to the matrix Λ and

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{3}$$

is the Lorentz Minkowski metric .

4 Lie Derivative

Let v be a vector field on a smooth manifold M and ϕ_v is a local flow generated for each $t \in R$. The map ϕ_v is diffeomorephsim of M and so it induces the movement functions such as push-forward and pull back.

Definition: The Lie derivative of the function f with respect to v is defined by:

$$L_v f = \lim_{t\to 0}\left(\frac{\phi_t^* f - f}{t}\right) = \frac{d}{dt}\phi_t^* f|_{t=0} \tag{4}$$

Lemma: since$\phi_t^* = f \circ \phi_t$, then

$$\frac{d}{dt}\phi_t^* f|_{t=0}\,(p) = \frac{d}{dt} f\big(\phi_t(p)\big)|_{t=0} = \frac{d}{dt} f\left(\gamma_p(t)\right)|_{t=0} = v(p).f \quad ,\forall p \in M \tag{5}$$

Where the tangent vector to γ_p at $p$is $X(p)$ concludes that $L_v f = vf$.

Although, if U, V be two vector fields on M. The Lie derivative of V with respect to U is given by:

$$L_U V = \lim_{t\to 0}\left(\frac{\phi_t^* V - V}{t}\right) = \frac{d}{dt}\phi_t^* V|_{t=0} \tag{6}$$

Where $\phi_t$is generated by U.

■

On the other hand, the Lie derivative is a *linear operator* which satisfies *Lebntiz identity* and always linearly dependents.

In this paper, however, the following special cases are needed to achieve the main goal.

Lie derivative of the Lorentzian metric

Definition : Let g_{ij}be the metric of Minkowski spacetime defined above (3), we find

$$L_v g_{ij} = g_{ab},cV^c + g_{ac}V^c,b + g_{cb}\,V^c,a \tag{7}$$

The first term of (7) seems to be normal derivative of a constant (the metric), so the equation (7) becomes

$$L_v g_{ij} = g_{ac}V^c,b + g_{cb}\,V^c,a \tag{8}$$

If we rotate or boost to save isometric of this change, we always have:

$$L_v g_{ij} = 0 \tag{9}$$

The Lie derivative of the metric plays a key role in the theory of Killing fields (For a review, see Carroll, 1997.p120), which are generators of continuous isometries. A vector field is a Killing field if the Lie derivative of the metric with respect to this field vanishes. I.e the vector v in (8) is so called Killing vector fields.

Furthermore, on solution (8) to be equal zero and according to Saad , in three dimensional Minkowskian space, the generators of matrices of rotations under this space are given by :

$x\frac{d}{dy} - y\frac{d}{dx}$, is called generator of rotation , and $x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}$ and $y\frac{\partial}{\partial t} + t\frac{\partial}{\partial y}$ are called generator of boost .

However, for the null (lightlike) generator we can use $(t-y)\frac{\partial}{\partial x} + x(\frac{\partial}{\partial y} + \frac{d}{dt})$. This suggestion is found in (Hall,2004; Saad,2016).

As a result, we use only rotation, one boost and the null generator for the null case. In the following section we solve the above generators as a system of ordinary differential equations to create matrices of rotation explicitly.

5 Results

The main conclusion of this paper is that matrices of rotation generate explicitly. Now we divide the discussion into three cases of study, i.e. rotation , boost and null case.

5.1 Spatial Rotation

It is important to begin with the most obvious analogue of rotation. The infinitesimal generator of this case is

$$x\, d/dy - y\, d/dx \tag{10}$$

Therefore the Killing vector field becomes :

$$V = (-y \quad x \quad 0)^T \tag{11}$$

This defines the 3×3 matrix in Minkowsks spacetime corresponding to the generator in (10), which is in (x, y, t) coordination by

$$L_k = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (12)$$

This matrix L_k is the matrix corresponding to infinitesimal generator about t-axis.

Now we have one parameter group of homomorphism $\dot{\psi}_s = L_k\, \psi_s$, so, $\psi_s(x) = e^{sL}\, x$. As x is a vector in E_1^3.

Our job now is to solve the system of ordinary differential equation of the form of :

$$\dot{\psi}_s = L_k\, \psi_s.$$

Then, to find eigenvalues let $\begin{vmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0$

$$therefore \quad \Rightarrow \quad \lambda_1 = 0\,, \lambda_2 = i, \lambda_3 = -i$$

Calculating the matrix exponential gives:

$$X(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (13)$$

This means any point $(0,0,t)$ is fixed. Also the axis of rotation is t-axis. Additionally, the orbit of any point of a spacelike curve with t-constant in this case is a circle centered with the origin.

5.2 Boost in direction of spacelike axis.

In this section the one parameter group of transformation which fixes each point in space-like is sought. Let Y- axis is the axis of rotation, and use

$$x\, \partial/\partial t + t\, \partial/\partial x \qquad (14)$$

Therefore the Killing vector field becomes :

$$\mathrm{V} = (t \quad 0 \quad x)^{\mathrm{T}} \qquad (15)$$

This defines the 3×3 matrix in Minkowksi spacetime which is similar to the generator in (14), which is in (x, y, t) coordination by

$$L_k = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad (16)$$

With the same calculation and discussion above, the matrix exponential gives:

$$X(\theta) = \begin{bmatrix} \cosh(\theta) & 0 & \sinh(\theta) \\ 0 & 1 & 0 \\ \sinh(\theta) & 0 & \cosh(\theta) \end{bmatrix} \qquad (17)$$

In this case the orbit of any point has fixed y-coordinate, discussion of the orbit is hyperbola-timelike. (lift for future work).

5.3 Null (lightlike) rotation in Minkowski spacetime.

Finally, we consider the situation where the axis of rotation is a null line. It is located in yt- plane i.e.(y=t) as given above before this section, so the generator of this case is:

$$(t - y)\partial/\partial x + x(\partial/\partial y + d/dt) \qquad (18)$$

Therefore the Killing vector field becomes :

$$\mathrm{V} = (t - y \quad x \quad x)^{\mathrm{T}} \qquad (19)$$

This defines the 3×3 matrix in Minkowksi spacetime corresponding to the generator in (10), which is in (x, y, t) coordination by

$$L_k = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad (20)$$

Same procedure, as before, of solving system of ODE.

Let

$$\begin{vmatrix} -\lambda & -1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0,$$

this gives $\lambda_{1,2,3} = 0$ "multiple eigenvalues "

At this time, with mansion to Perko, the general solution to the linear system is given by

$$X(\theta) = e^{\lambda\theta}\left[I + N\theta + \cdots + \frac{N^k\theta^k}{k!}\right] \qquad (21)$$

Where $N = L - S$ is an $n \times n$ matrix is said to be nilpotent of order k if $\mathrm{N}^{k-1} \neq 0$ and $\mathrm{N}^k = 0$, $k \leq n$, $S = diag(\lambda)$, L is a matrix called Lorentz transformation .

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ N = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$N^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \text{ , } N^3 = 0 \tag{22}$$

$$X(\theta) = e^{\lambda\theta}\left[I + N\theta + \frac{1}{2} N^2\theta^2\right] \tag{23}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -\theta & \theta \\ \theta & 0 & 0 \\ \theta & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\theta^2 & \theta^2 \\ 0 & -\theta^2 & \theta^2 \end{bmatrix}$$

Therefore, the one parameter subgroups of rotation matrices is

$$\mathrm{X}(\theta) = \begin{bmatrix} 1 & -\theta & \theta \\ \theta & 1 - \frac{\theta^2}{2} & \frac{\theta^2}{2} \\ \theta & -\frac{\theta^2}{2} & 1 + \frac{\theta^2}{2} \end{bmatrix} \tag{24}$$

This one parameter group fixes line $y = t \ in \ yt -$ plane

Therefore, the axis of rotation is given by $\mathrm{l} = (0,1,1)$. And the orbit seems to be a parabola-time, but the discussion on is out of our scope at the moment.

In summary, the aforementioned three cases of matrices of rotation are called in physics as one parameter groups of isometry that leaves line fixed. These are the main results of this paper.

6 Conclusions

This research shows that the Lie derivative and Killing vector field are used explicitly to generate three rotational matrices (space, boost, and null). The main findings revealed that computational generators find the matrices of rotation assuming the axis of rotation is dependent on a vector field space, time, or null.

Further studies are in need to examine the geometry of surfaces in Minkowskian spaces. All geometrical aspects of curvatures, Gaussian maps, and mean curvatures are in parallel with these matrices.

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References

Pressley, A. N. (2010). Elementary differential geometry. Springer Science & Business Media.

Hall, G. S. (2004). Symmetries and curvature structure in general relativity (Vol. 46). world scientific.

Hilgert, J., & Neeb, K. H. (2011). Structure and geometry of Lie groups. Springer Science & Business Media.

Lopez ,R . (2014) Differential Geometry of Curves and Surfaces in Lorentz - Minkowski space, International Electronic Journal of Geometry, 7(2014), 44-107 7 (1) , 44-107 . DOI: 10.36890/iejg.594497

Saad ,A . (2016) On Calculus of Manifolds with Special Emphasis of 3D Minkowski Space $\mathrm{M}^{2,1}$ International Journal of Engineering and Applied Sciences, 3(1), 257742.

Carroll, S. M. (1997). Lecture notes on general relativity. arXiv preprint gr-qc/9712019.

Perko ,L . (2000) " Differential Equations and Dynamical Systems", third edition , Springer-Verlag New York Berlin Heidelberg , Germany.

Torres del Castillo G.F. (2012)" Lie Derivatives In Differentiable Manifolds", Springer, Birkhäuser, Boston,USA.

Duggal ,K and Sharma ,R. (1999) "Symmetries of Spacetimes and Riemannian Manifolds " (Vol.487) , Springer Science Business Media

Saad, A. I., Muhammed, S. A. E., & Elmabrouk, T. S (2016) On Geodesics of 3D Surfaces of Rotations in Euclidean and Minkowskian Spaces." International Journal of Engineering and Applied Sciences, 3(1), 257742.

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