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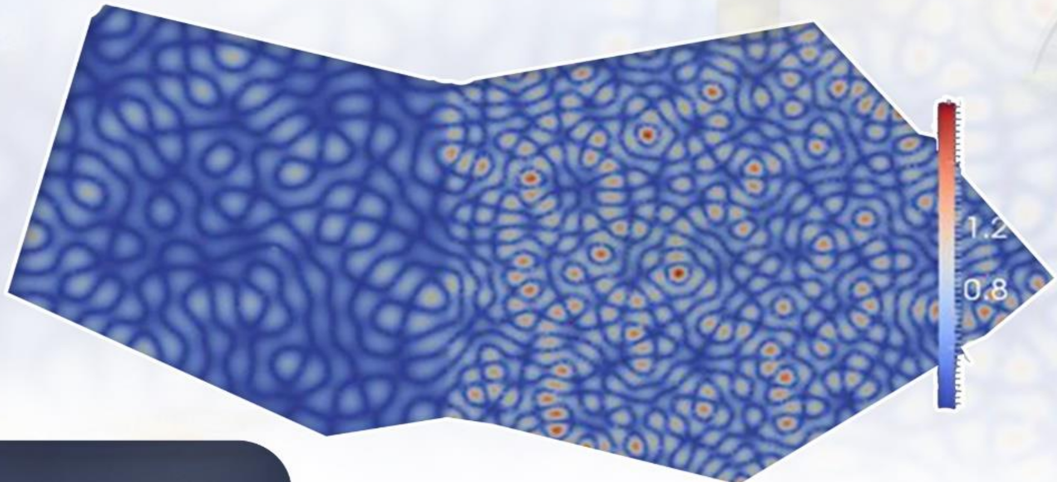

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Using Anuj Transform to Solve Ordinary Differential Equations with Variable Coefficients

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A B S T R A C T

In this paper, a new integral transform was applied to find the solution of ordinary differential equations with variable coefficients because of its importance in various fields of science, especially mathematics. This integral transform is called Anuj transform. The Anuj transform is one of the modern integral transforms that is effective in solving ordinary differential equations in general and ordinary differential equations with variable coefficients in particular. In order to achieve this; and based on a set of relations that have been included in this paper, we have implemented a number of methods to reach important formulas and results. These results have been applied to different set of problems of ordinary differential equations with variable coefficients. The solutions of those problems were reached in a smooth manner, based on some basics of ordinary differential equations which give a simple and clear idea of using the Anuj transform to solve ordinary differential equations with variable coefficients.

1 Introduction

Ordinary differential equations (ODE) are one of the most important areas of mathematics science in general, applied mathematics in particular. Ordinary differential equations have many types and one of them, not limited to, is ordinary differential equations with variable coefficients. These days, ordinary differential equations with variable coefficients are widely used in astronomy, physics and engineering mathematics (Aggarwal et al., 2018). Occasionally, to solve like these equations, the calculations may be very complicated and ultimately frustrating. Integral transforms play a big role in solving such equations (M. & Mahgoub, 2019, 2017 & 2016), (Elzaki et al., 2012). Many papers relevant have been recently published carried out to solve ordinary differential equations with variable coefficients using integral transforms, for instance, Sumudu transform, Kamal transform and Mohand transform, to name a few. In general, integral transforms techniques are not useful for most problems. However, integral transforms have become an important tool to deal with problems in applied mathematics, theoretical mechanics, statistics, mathematical physics and pharmacokinetics (M. & Mahgoub, 2017), (Devi et al., 2017). "Integral transform method is widely used to solve the several differential equations with the initial values or boundary conditions" (Devi et al., 2017). The great importance of such transforms appears in solving this type of ordinary differential equations without much complexity. Latterly, a new transform has appeared which named Anuj transform. The definition of Anuj transform as mentioned by Kumar et al. (2021) of a piecewise continuous exponential order function $\psi(t); t > 0$ is defined as the following integral equation:

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$$\Lambda\{\psi(t)\} = \nu^2 \int_0^\infty \psi(t) e^{-\frac{t}{\nu}}\, dt = \Psi(\nu)\,, \nu > 0 \qquad (\mathbf{1})$$

where ν is real parameter and Λ is the Anuj transform operator, This paper will deduce Anuj transform formulas for some functions such as $t\psi(t), t^2\psi(t)$ & $t\psi'(t)$, which enable us to extract any formulas of these kinds.

Therefore, the purpose of this paper will show the applicability and efficiency of Anuj transform to solve some ordinary differential equations with variable coefficients.

1 Anuj Transform and Inverse Anuj Transform of Some Functions

The following Anuj transform and inverse Anuj transform of some functions are summarized in tables (1) & (2) (Kumar et al., 2021). Furthermore, these formulas are adapted from the same reference.

Table (1). Anuj transform of some functions.

S. No.	$\psi(t)$	$\Lambda\{\psi(t)\} = \Psi(\nu)$
1	1	ν^3
2	t	ν^4
3	t^2	$2!\,\nu^5$
4	$t^m\,; m \in \mathbb{N}$	$m!\,\nu^{m+3}$
5	$t^m\,;$ $m > -1$	$\Gamma(m+1)\nu^{m+3}$
6	e^{at}	$\frac{\nu^3}{1-a\nu}$
7	$\sin\beta t$	$\frac{\beta\nu^4}{1+\beta^2\nu^2}$
8	$\cos\beta t$	$\frac{\nu^3}{1+\beta^2\nu^2}$
9	$\sinh\beta t$	$\frac{\beta\nu^4}{1-\beta^2\nu^2}$
10	$\cosh\beta t$	$\frac{\nu^3}{1-\beta^2\nu^2}$
11	$e^{at}\psi(t)$	$(1-ap)^2\Psi\left(\frac{\nu}{1-a\nu}\right)$

Table (2). Inverse Anuj transform of some functions.

S. No.	$\Psi(\nu)$	$\psi(t) = \Lambda^{-1}\{\Psi(\nu)\}$
1	ν^3	1
2	ν^4	t
3	ν^5	$\frac{t^2}{2!}$
4	$\nu^m\,;\ m \in \mathbb{N}$	$\frac{t^{m-3}}{(m-3)!}$
5	ν^{m+3}; $m > -1$	$\frac{t^m}{\Gamma(m+1)}$
6	$\frac{\nu^3}{1-a\nu}$	e^{at}
7	$\frac{\beta\nu^4}{1+\beta^2\nu^2}$	$\frac{\sin\beta t}{\beta}$
8	$\frac{\nu^3}{1+\beta^2\nu^2}$	$\cos\beta t$
9	$\frac{\nu^4}{1-\beta^2\nu^2}$	$\frac{\sinh\beta t}{\beta}$
10	$\frac{\nu^3}{1-\beta^2\nu^2}$	$\cosh\beta t$

2 Anuj Transform of Derivatives of Some Functions (Kumar et al., 2021)

$$\Lambda\{\,\psi'(t)\,\} = \frac{1}{\nu}\Psi(\nu) - \nu^2\psi(0) \qquad (\mathbf{2})$$

$$\Lambda\{\psi''(t)\,\} = \frac{1}{\nu^2}\Psi(\nu) - \nu\psi(0) - \nu^2\frac{d}{d\nu}\psi(0) \qquad (\mathbf{3})$$

$$\Lambda\{\psi'''(t)\,\} = \frac{1}{\nu^3}\Psi(\nu) - \psi(0) - \nu\frac{d}{d\nu}\psi(0) - \nu^2\frac{d^2}{d\nu^2}\psi(0) \ (\mathbf{4})$$

3 Anuj Transform of $t\psi(t)$ & $t^2\psi(t)$:

If $\boldsymbol{\Lambda\{\psi(t)\} = \Psi(\nu)}$, then:

i. $\Lambda\{t\psi(t)\} = \nu^2\frac{d}{d\nu}\Psi(\nu) - 2\nu\Psi(\nu)$.

Proof. Since, $\Lambda\{\psi(t)\} = \nu^2\int_0^\infty \psi(t)e^{-\frac{t}{\nu}}dt = \Psi(\nu)$

$$\therefore \frac{d}{d\nu}\Psi(\nu) = \frac{d}{d\nu}\left(\nu^2\int_0^\infty \psi(t)e^{-\frac{t}{\nu}}dt\right)$$

$$= \nu^2\int_0^\infty \frac{t}{\nu^2}\psi(t)e^{-\frac{t}{\nu}}dt + 2\nu\int_0^\infty \psi(t)e^{-\frac{t}{\nu}}dt$$

$$= \frac{1}{\nu^2}\Lambda\{t\psi(t)\} + \frac{2}{\nu}\Psi(\nu)$$

$$\Rightarrow \Lambda\{t\psi(t)\} = \nu^2\frac{d}{d\nu}\Psi(\nu) - 2\nu\Psi(\nu).\ (\mathbf{5})$$

ii. $\Lambda\{t^2\psi(t)\} = \nu^2\left[\nu^2\frac{d^2}{d\nu^2}\Psi(\nu) - 2\nu\frac{d}{dp}\Psi(\nu) + 2\Psi(\nu)\right]$

Proof. Since, $\Lambda\{t\psi(t)\} = \nu^2\frac{d}{d\nu}\Psi(\nu) - 2\nu\Psi(\nu)$, so putting $t\psi(t)$ instead of $\psi(t)$ yields:

$$\Lambda\{t^2\psi(t)\} = \nu^2\frac{d}{d\nu}\Lambda\{t\psi(t)\} - 2\nu\Lambda\{t\psi(t)\}$$
$$= \nu^2\frac{d}{d\nu}\left[\nu^2\frac{d}{d\nu}\Psi(\nu) - 2\nu\Psi(\nu)\right] - 2\nu\left[\nu^2\frac{d}{d\nu}\Psi(\nu) - 2\nu\Psi(\nu)\right]$$
$$= \nu^4\frac{d^2}{d\nu^2}\Psi(\nu) + 2\nu^3\frac{d}{d\nu}\Psi(\nu) - 2\nu^3\frac{d}{d\nu}\Psi(\nu) - 2\nu^2\Psi(\nu) - 2\nu^3\frac{d}{d\nu}\Psi(\nu) + 4\nu^2\Psi(\nu)$$
$$\therefore\ \Lambda\{t^2\psi(t)\} = \nu^2\left[\nu^2\frac{d^2}{d\nu^2}\Psi(\nu) - 2\nu\frac{d}{d\nu}\Psi(\nu) + 2\Psi(\nu)\right].(\mathbf{6})$$

4 Anuj Transform of $t\psi'(t)$ & $t^2\psi'(t)$:

If $\boldsymbol{\Lambda\{\psi(t)\} = \Psi(\nu)}$, then:

i. $\Lambda\{t\psi'(t)\} = \nu\frac{d}{d\nu}\Psi(\nu) - 3\Psi(\nu) - \nu^4\frac{d}{d\nu}\psi(0)$

Proof. Since, $\Lambda\{\psi'(t)\} = \frac{1}{\nu}\Psi(\nu) - \nu^2\psi(0)$, so in (**5**) put $\psi'(t)$ instead of $\psi(t)$. Therefore:

$$\Lambda\{t\psi'(t)\} = \nu^2\frac{d}{d\nu}\Lambda\{\psi'(t)\} - 2\nu\Lambda\{\psi'(t)\}$$
$$= \nu^2\frac{d}{d\nu}\left[\frac{1}{\nu}\Psi(\nu) - \nu^2\psi(0)\right] - 2\nu\left[\frac{1}{\nu}\Psi(\nu) - \nu^2\psi(0)\right]$$
$$= \nu\frac{d}{d\nu}\Psi(\nu) - \Psi(\nu) - \nu^4\frac{d}{d\nu}\psi(0) - 2\nu^3\psi(0) - 2\Psi(\nu) + 2\nu^3\psi(0)$$
$$\therefore\ \Lambda\{t\psi'(t)\} = \nu\frac{d}{d\nu}\Psi(\nu) - 3\Psi(\nu) - \nu^4\frac{d}{d\nu}\psi(0)\ (\mathbf{7})$$

ii. $\Lambda\{t^2\psi'(t)\} = \nu\left[\nu^2\frac{d^2}{d\nu^2}\Psi(\nu) - 4\nu\frac{d}{d\nu}\Psi(\nu) + 6\Psi(\nu)\right] - \nu^4\frac{d}{d\nu}\left[\nu^2\frac{d}{d\nu}\psi(0)\right]$

Proof. As before, we get:

$$\Lambda\{t^2\psi'(t)\} = \nu^2\frac{d}{d\nu}\Lambda\{t\psi'(t)\} - 2\nu\Lambda\{t\psi'(t)\}$$
$$= \nu^2\frac{d}{d\nu}\left[\nu\frac{d}{d\nu}\Psi(\nu) - 3\Psi(\nu) - \nu^4\frac{d}{d\nu}\psi(0)\right] - 2\nu\left[\nu\frac{d}{d\nu}\Psi(\nu) - 3\Psi(\nu) - \nu^4\frac{d}{d\nu}\psi(0)\right]$$
$$= \nu^3\frac{d^2}{d\nu^2}\Psi(\nu) + \nu^2\frac{d}{d\nu}\Psi(\nu) - 3\nu^2\frac{d}{d\nu}\Psi(\nu) - \nu^6\frac{d^2}{d\nu^2}\psi(0) - 4\nu^5\frac{d}{d\nu}\psi(0) - 2\nu^2\frac{d}{d\nu}\Psi(\nu) + 6\nu\Psi(\nu) + 2\nu^5\frac{d}{d\nu}\psi(0)$$
$$= \nu^3\frac{d^2}{d\nu^2}\Psi(\nu) - 4\nu^2\frac{d}{d\nu}\Psi(\nu) + 6\nu\Psi(\nu) - \nu^6\frac{d^2}{d\nu^2}\psi(0) - 2\nu^5\frac{d}{d\nu}\psi(0)$$
$$\therefore \Lambda\{t^2\psi'(t)\} = \nu\left[\nu^2\frac{d^2}{d\nu^2}\Psi(\nu) - 4\nu\frac{d}{d\nu}\Psi(\nu) + 6\Psi(\nu)\right] - \nu^4\frac{d}{d\nu}\left[\nu^2\frac{d}{d\nu}\psi(0)\right]\ (\mathbf{8})$$

Thus, we can deduce the following relations:

$$\Lambda\{t\psi''(t)\} = \frac{d}{d\nu}\Psi(\nu) - \frac{4}{\nu}\Psi(\nu) + \nu^2\psi(0) - \nu^3\frac{d}{d\nu}\left[\nu\frac{d}{d\nu}\psi(0)\right].(\mathbf{9})$$
$$\Lambda\{t^2\psi''(t)\} = \nu^2\frac{d^2}{d\nu^2}\Psi(\nu) - 6\nu\frac{d}{d\nu}\Psi(\nu) + 12\Psi(\nu) - \nu^3\frac{d}{d\nu}\left[\nu^3\frac{d^2}{d\nu^2}\psi(0)\right].(\mathbf{10})$$

Notice: Anuj Transform of functions $\{t^n\psi^{(n)}(t)\}(;\ n \in \mathbb{N})$ can be calculated as the way posed in the last paragraphs.

5 APPLICATIONS

In this section, we will provide some different examples:

Example (1): Solve the differential equation:

$$t^2y'' - ty' + y = 5, \text{ with } [y(0) = 5\ \&\ y'(0) = 3].$$

Solution: Applying the Anuj transform to both sides of the given equation, we get:

$$\Lambda\{t^2y''\} - \Lambda\{ty'\} + 6\Lambda\{y\} = \Lambda\{5\}.$$

$$\nu^2\frac{d^2}{d\nu^2}\Lambda\{y\} - 6\nu\frac{d}{d\nu}\Lambda\{y\} + 12\Lambda\{y\} - \nu^3\frac{d}{d\nu}\left[\nu^3\frac{d^2}{d\nu^2}y(0)\right] - \nu\frac{d}{d\nu}\Lambda\{y\} + 3\Lambda\{y\} + \nu^4\frac{d}{d\nu}y(0) + \Lambda\{y\} = \Lambda\{5\}.$$

$$\nu^2\frac{d^2}{d\nu^2}\Lambda\{y\} - 7\nu\frac{d}{d\nu}\Lambda\{y\} + 16\Lambda\{y\} = 5\nu^3.$$

Let $M = \Lambda\{y\}$

$$\Rightarrow \nu^2\frac{d^2M}{d\nu^2} - 7\nu\frac{dM}{d\nu} + 16M = 5\nu^3$$

which is the Cauchy-Euler differential equation. So, we will consider:

$$\nu = e^x \Rightarrow x = \ln\nu\ \Rightarrow\ \frac{d^2M}{d\nu^2} = e^{-2x}\left(\frac{d^2M}{dx^2} - \frac{dM}{dx}\right), \frac{dM}{d\nu} = e^{-x}\frac{dM}{dx}.$$

Thus, by substituting above work into the last equation, we get

$e^{2x}e^{-2x}\left(\frac{d^2M}{dx^2}-\frac{dM}{dx}\right)-7e^{x}e^{-x}\frac{dM}{dx}+16M=5e^{3x}$

$\Rightarrow M''-8M'+16M=5e^{3x}$

$\Rightarrow M_c=C_1e^{4x}+C_2xe^{4x}\Rightarrow M_c=C_1v^4+C_2v^4\ln v$

& $M_p=\frac{5}{D^2-8D+16}e^{3x}\ \Rightarrow M_p=5e^{3x}\Rightarrow M_p=5v^3$

$\therefore\ M_g=C_1v^4+C_2v^4\ln v+5v^3.$

Here, note that if we require $y(0)$ to be finite, we are forced to conclude that $C_2=0$.

$\therefore\ \Lambda\{y\}=C_1v^4+5v^3\Rightarrow y=\Lambda^{-1}\{C_1v^4\}+\Lambda^{-1}\{5v^3\}\Rightarrow y=C_1t+5$

$\because y'(0)=3\Rightarrow\ y'=C_1\Rightarrow C_1=3.$

$\therefore\ y(t)=3t+5.$

Example (2): Solve the differential equation:

$$ty''-ty'-y=0$$

with the initial condition $y(0)=0$ & $y'(0)=2$.

Solution: Taking Anuj transform to both sides of given equation to give us:

$\Lambda\{ty''\}-\Lambda\{ty'\}-\Lambda\{y\}=0.$

$\Rightarrow\frac{d}{dv}\Lambda\{y\}-\frac{4}{v}\Lambda\{y\}+v^2y(0)-v^3\frac{d}{dv}\left[v\frac{d}{dv}y(0)\right]-v\frac{d}{dv}\Lambda\{y\}+3\Lambda\{y\}+v^4\frac{d}{dv}y(0)-\Lambda\{y\}=0.$

$(1-v)\frac{d}{dv}\Lambda\{y\}-\left(\frac{4}{v}+4\right)\Lambda\{y\}=0.$

$\Rightarrow\int\frac{d\Lambda\{y\}}{\Lambda\{y\}}=\int\frac{4(1+v)}{v(1-v)}dv\Rightarrow\ln(\Lambda\{y\})=4\ln v-2\ln(1-v)+\ln C$

$\Rightarrow\frac{(1-v)^2}{v^4}\Lambda\{y\}=C\ \Rightarrow\ \Lambda\{y\}=C\frac{v^4}{(1-v)^2}.$

Now, by applying inverse Anuj transform, we get

$y=\Lambda^{-1}\left\{C\frac{v^4}{(1-v)^2}\right\}\Rightarrow y=Cte^t.$

$$\because\ y'(0)=2\ \Rightarrow 2=C\ \Rightarrow C=2$$

$$\Rightarrow\ y(t)=2te^t.$$

Example (3): (Attaweel & Almassry 2020)

Solve the differential equation:

$$ty''-y'=t^2$$

with the initial condition $y(0)=0$ & $y'(0)=0$.

Solution: Applying Anuj transform to both sides of given equation which give us:

$\Lambda\{ty''\}-\Lambda\{y'\}=\Lambda\{t^2\}.$

$\Rightarrow\frac{d}{dv}\Lambda\{y\}-\frac{4}{v}\Lambda\{y\}+v^2y(0)-v^3\frac{d}{dv}\left[v\frac{d}{dv}y(0)\right]-\frac{1}{v}\Lambda\{y\}+v^2y(0)=\Lambda\{t^2\}$

$\frac{d}{dv}\Lambda\{y\}-\frac{5}{v}\Lambda\{y\}=2v^5$

which is a linear differential equation and it has the integrative factor: $\lambda=\frac{1}{v^5}$.

$\Rightarrow\frac{1}{v^5}\Lambda\{y\}=2v+C\Rightarrow\Lambda\{y\}=2v^6+Cv^5$

Now applying inverse Anuj transform, we get

$y(t)=\frac{t^3}{3}+\frac{Ct^2}{2}.$

Example (4): (Aggarwal et al., 2018)

Solve the differential equation:

$ty'-2y=0$, with $y(0)=0$.

Solution: Applying the Anuj transform to both sides of the given equation, we get:

$\Lambda\{ty'\}-2\Lambda\{y\}=0.$

$v\frac{d}{dv}\Lambda\{y\}-3\Lambda\{y\}-v^4\frac{d}{dv}y(0)-2\Lambda\{y\}=0.$

$\Rightarrow\frac{d}{dv}\Lambda\{y\}=\frac{5}{v}\Lambda\{y\}$. Let $M=\Lambda\{y\}$

$\Rightarrow\ \frac{dM}{dv}=\frac{5}{v}M\ \Rightarrow\ \frac{dM}{M}=5\frac{dv}{v}$

$\Rightarrow\ M=Cv^5\Rightarrow\Lambda\{y\}=Cv^5$

$\Rightarrow\ y=\Lambda^{-1}\{Cv^5\}$

$\therefore\ y(t)=C\frac{t^2}{2}.$

6 Conclusions

In this paper, the researcher has discussed the application of Anuj transform to solve ODE's with variable coefficients by obtain Anuj transform formulas of functions: $t\psi(t), t^2\psi(t), t\psi'(t), t^2\psi'(t), t\psi''(t)$ & $t^2\psi''(t)$, and then apply those formulas in some problems. The results of the present study show that the Anuj transform is very useful integral transform for solving such equations. In addition, all the obtained solutions of the indicated problems are satisfied by putting them back in the corresponding equations. In future, Anuj transform can be used to solve differential equations more broadly.

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References

Aggarwal, Sudhanshu., Sharma, Nidhi., Chauhan, Raman., Gupta, Anjana. & Khandelwal. Astha (2018). *A New Application of Mahgoub Transform for Solving Linear Ordinary Differential Equations with Variable Coefficients.* 9(6): 520-525. Journal of Computer and Mathematical Sciences.

Attaweel, M. & Almassry, Haneen (2020). *On the Mohand Transform and Ordinary Differential Equations with Variable Coefficients.* 35(1): 1-6. Libya: Al-Mukhtar Journal of Sciences.

Devi, Anuj., Roy, Priyanka., & Gill, Vinod (2017). *Solution of Ordinary Differential Equations with Variable Coefficients Using Elzaki Transform*. 1(9): 186-194. Asian Journal of Applied Science and Technology (AJAST).

Elzaki Tariq, Elzaki Salih, Hilal Eman (2012). *Elzaki and Sumudu Transforms for Solving Some Differential Equations.* 8(2): 167-173. India: Global Journal of Pure and Applied Mathematical.

Kumar Anuj, Bansal Shikha, Aggarwal Sudhanshu (2021). *A new novel integral transform "Anuj transform" with application*. (9): 12741-12751. Design Engineering.

M. Mohand, Mahgoub A. (2019). *The new integral transform "Sawi Transform"*. 11(14): 81-87. Advances in Theoretical and Applied Mathematics.

M. Mohand, Mahgoub A. (2017). *The new integral transform "Mohand Transform".* 11(12): 113-120. Advances in Theoretical and Applied Mathematics.

M. Mohand, Mahgoub A. (2016). *The new integral transform "Mahgoub Transform".* 11(4): 391-398. Advances in Theoretical and Applied Mathematics.

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