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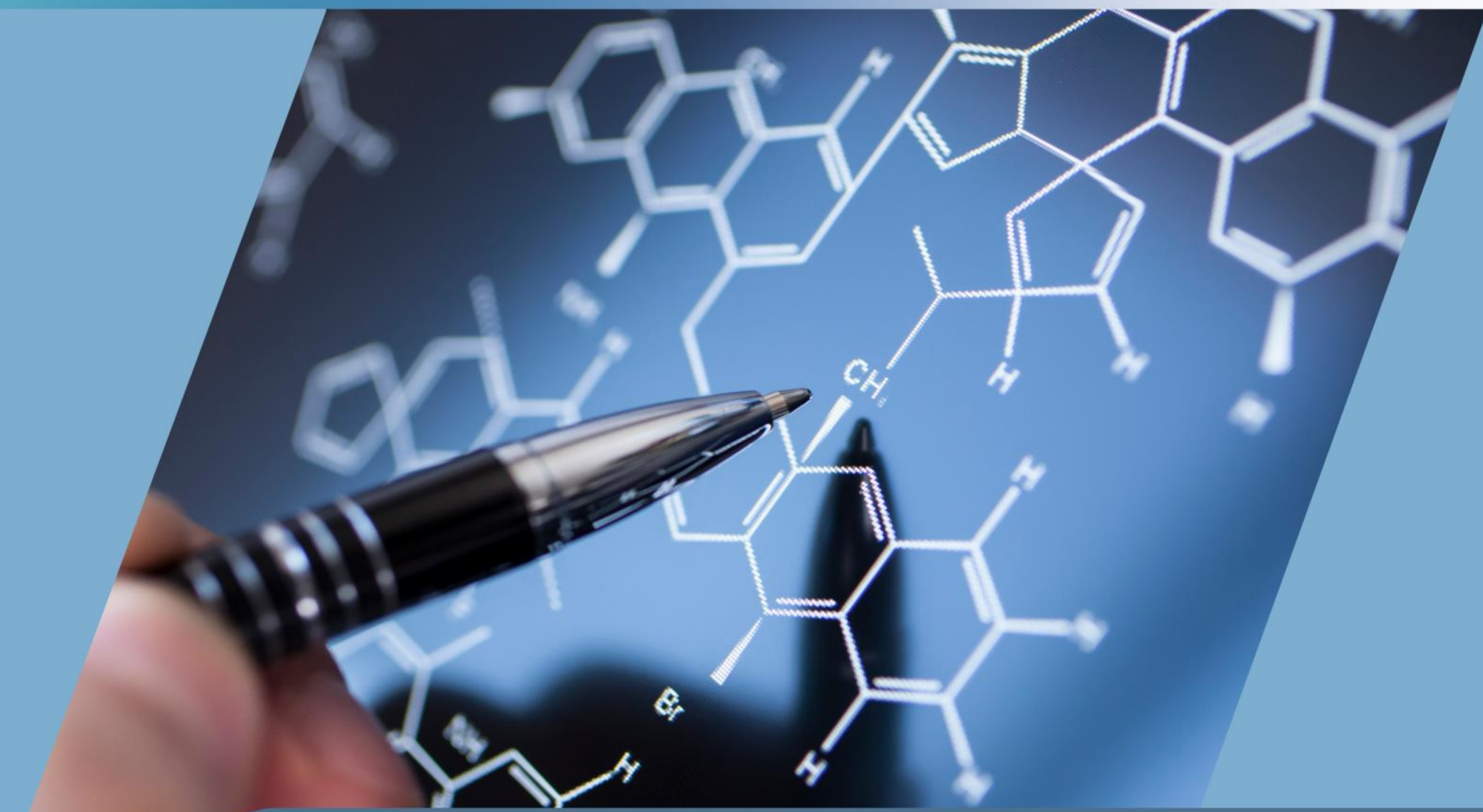

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Iterative Processes Methods for Solving Boundary Value Problem for the Caputo Fractional Differential Equations

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A B S T R A C T

In this article, we introduce a study of approximate solutions for the Caputo fractional differential equations with boundary conditions in Banach space. We transformed given equations into equivalent integral equations for the construction of contraction mapping and other compact mapping, both of which allow for the proof of the existence of a solution. The ultimate goal of this study is to present a comparison of the speed convergence of the approximate solutions of Caputo Fractional differential equations obtained by using the processes repetitiveness of the Picard, Mann, Picard-Mann hybrid, Picard-Krasnoselskii hybrid, and Ishikawa methods to the general solutions.

1 Introduction

The study of the iterative processes for fractional differential equations to find approximating solutions is an active area of research. However, the exact or approximate analytical solutions are preferable for the boundary value problems because they permit the investigation of the qualitative properties of the appropriate control systems. The Picard iterative method is one of the simplest iteration methods to approximate the solution of fixed-point equations for fractional differential equations involving nonlinear contractive operators.

(Chidume et al., 2013; Chidume, 2014) introduced a study on the Picard iteration process and Krasnoselskii-type sequences. (Berinde, 2004) introduced a study on convergence for iterative methods and showed that the Picard iteration converges more rapidly than the Mann iteration. (Berinde et al., 2005) proved the Krasnoselskij iteration was speedily convergent for approximating fixed points in Banach spaces. The authors (Abdeljawad et al., 2020; Ghiura, 2021) published a study on the speed of convergence for iterative methods and explained in their results whether the convergence is strong or weak for a class of mappings. The existence and uniqueness of the solutions for the nonlinear fractional differential equation boundary value problem considered by the authors (Furati & Tatar, 2005; Lyons et al., 2017; Wang et al., 2014; Zhang, 2006).

2 Preliminaries

Let A be a real Banach space, and $M \neq \phi$ be a convex subset of A. Let $F : M \to M$ be a mapping, then F is said to be non-expansive mapping if:

$$\|Fx_1 - Fx_2\| \leq \|x_1 - x_2\| \quad \forall\ x_1, x_2 \in M \tag{2.1}$$

and contraction mapping if:

$$\|Fx_1 - Fx_2\| \leq \rho\|x_1 - x_2\| \quad \forall\ x_1, x_2 \in M,\ \rho \in (0,1) \tag{2.2}$$

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Now, we define the Picard method (Picard, 1890) as the follows, let $x_0 \in M$ and $\{x_i\} \subset M$

$$x_{i+1} = Fx_i, \qquad i = 0,1,... \qquad (2.3)$$

Let $\{\eta_i\}_{i=0}^{\infty}$ be a sequence in $(0,1)$ for $u_0 \in M$, then the Mann iterative method defined by the sequence $\{u_i\} \subset M$ (Mann, 1953):

$$u_{i+1} = (1-\eta_i)u_i + \eta_i Fu_i \quad ,i = 0,1,... \qquad (2.4)$$

Let $\{y_i\} \subset M$, the Krasnoselskii iteration method defined by (Krasnoselskii, 1955):

$$y_{i+1} = (1-\kappa)y_i + \kappa Fy_i \quad ,i = 0,1,... \qquad (2.5)$$

where $\kappa \in (0,1)$ is real a constant and $y_0 \in M$.

(Khan, 2013) presented the Picard-Mann hybrid method, which is given by the sequence $\{v_i\} \subset M$ as follows:

$$v_{i+1} = Fy_i, \qquad (2.6)$$
$$y_i = (1-\eta_i)v_i + \eta_i Fv_i \quad , \quad i = 0,1,...$$

where $\{\eta_i\}_{i=0}^{\infty}$ are appropriately chosen sequences in $(0,1)$ and $v_0 \in M$.

(Okeke & Abbas, 2017) introduced a study on the Picard-Krasnoselskii hybrid iterative process. This iterative process is given by the sequence $\{x'_i\} \subset M$ as follows:

$$x'_{i+1} = Fy_i, \qquad (2.7)$$
$$y_i = (1-\kappa)x_i + \kappa Fx_i \quad , \quad i = 0,1,...$$

where $\kappa \in (0,1)$ and $x_0 \in M$.

The Ishikawa iterative process is defined by the sequence $\{v_i\} \subset F$ as follows (Ishikawa, 1974):

$$w_{i+1} = (1-\eta_i)w_i + \eta_i Fy_i, \qquad (2.8)$$
$$y_i = (1-\xi_i)w_i + \xi_i Fw_i, \quad i = 0,1,...$$

where $\{\eta_i\}_{i=0}^{\infty}, \{\xi_i\}_{i=0}^{\infty}$ are appropriately chosen sequences in $(0,1)$ and $w_0 \in M$.

Definition 2.1. Let $\{a_i\}_{i=0}^{\infty}, \{b_i\}_{i=0}^{\infty} \in [0,\infty)$ converge to ℓ_1 and ℓ_2 respectively. Assume that there exists the following limit:

$$\lim_{i\to\infty}\left|\frac{a_i + \ell_1}{b_i + \ell_2}\right| = l$$

1- If $l = 0$, then a_i converges more rapidly to ℓ_1 than b_i to ℓ_2.
2- If $0 < l < \infty$, then a_i and b_i converge at the same rate.

Definition 2.2. Suppose that the iteration sequences $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ both converge to a fixed point q. Let $\{a_i\}, \{b_i\} \in \Re^+$, such that:

$$\|x_i - q\| \le a_i \quad , \forall\ i \in \aleph,$$
$$\|y_i - q\| \le b_i \quad , \forall\ i \in \aleph,$$

where a_i and b_i are converging to 0. If a_i converges faster than b_i, then $\{x_i\}$ is said to converge faster than $\{y_i\}$ to q.

Definition 2.3. If $\{x_i\}_{i=0}^{\infty}$ and $\{y_i\}_{i=0}^{\infty}$ are iteration sequences that converge to the unique fixed point q of F, then $\{x_i\}$ converges faster than $\{y_i\}$, if

$$\lim_{i\to\infty}\left\|\frac{x_i - q}{y_i - q}\right\| = 0$$

Lemma 2.1. (Şoltuz & Otrocol, 2007)

Let $\{s_i\}_{i=1}^{\infty} \in \Re^+$ which satisfies: $s_{i+1} \le (1-\delta_i)s_i$, if $\{\delta_i\}_{i=1}^{\infty} \in (0,1)$ and $\sum_{i=1}^{\infty}\delta_i = \infty$, then $s_i \to 0$ as $i \to \infty$.

Lemma 2.2. (Osilike & Aniagbosor, 2000)

Let $\{s_i\}_{i=1}^{\infty}, \{\delta_i\}_{i=1}^{\infty}$ and $\{\sigma_i\}_{i=1}^{\infty}$ be sequences in $\Re^+$ which satisfies the inequality:

$$s_{i+1} \le (1+\delta_i)s_{i+1} + \sigma_i \ , \qquad \forall\ i \ge 1. \qquad (2.9)$$

If $\sum_{i=1}^{\infty}\delta_i < \infty$ and $\sum_{i=1}^{\infty}\sigma_i < \infty$, then $s_i \to 0$ as $i \to \infty$.

3 Fractional calculus

Fractional calculus has operations of integration and differentiation of fractional order, and the theory of fractional differential equations has affected many authors in mathematics, physics, and engineering. In this section, we will review some fundamental definitions and lemmas concerning fractional calculus operators (Diethelm & Ford, 2010; Kilbas et al., 2006):

Definition 3.1. Let $\alpha > 0$ and $g : \Re^+ \to \Re$ be a function, the left and right RLF integral operators I_{a+}^{α} and I_{b-}^{α} of order $\alpha \in \Re^+$ of g are defined by:

$$I_{b-}^{\alpha} g(t) = D_{b+}^{-\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau - t)^{\alpha-1} g(\tau) d\tau, t \in [a,b) \subset \Re^+ \quad (3.1)$$

$$I_{a+}^{\alpha} g(t) = D_{a+}^{-\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} g(\tau) d\tau, t \in (a,b] \subset \Re^+ \quad (3.2)$$

respectively, where $\Gamma(.)$ is defined by $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt$.

Definition 3.2. Let $\alpha \in \Re^+$, $0 < \alpha < 1$, the left and right RLF derivative of g are defined by:

$$D_{a+}^{\alpha} g(t) := D I_{a+}^{1-\alpha} g(t), \quad \forall t \in (a,b] \quad (3.3)$$

$$D_{b-}^{\alpha} g(t) := -D I_{b-}^{1-\alpha} g(t), \quad \forall t \in [a,b) \quad (3.4)$$

The following properties are held for fractional integrals and derivatives:-

1- $D_{a^+}^{\nu} I_{a^+}^{\alpha} g(t) = I_{a^+}^{\alpha-\nu} g(t)$ and $D_{b^-}^{\nu} I_{b^-}^{\alpha} g(t) = I_{b^-}^{\alpha-\nu} g(t)$ if $\alpha, \nu \in \Re^+$.

2- $D^{-\alpha} D^{-\nu} g(t) = I^{\alpha+\nu} g(t) = D^{-\alpha-\nu} g(t), \ \alpha, \nu > 0.$

3- $I^{\alpha} I^{\nu} g(t) = I^{\nu} I^{\alpha} g(t)$

4- $I^{\alpha} D^{\alpha} g(t) = g(t), 0 < \alpha < 1, \ g \in C[a,b],$ and $D^{\alpha} g(t) \in C(a,b) \cap L(a,b).$

5- $I^{\alpha} t^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)} t^{\nu-\alpha} \quad , \alpha \geq 0, \nu > -1$

Definition 3.3. The Caputo derivative of order α for a function $g : \Re^+ \to \Re$ is given by:

$${}^{C}D_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(\upsilon - \alpha)} \int_0^t (t - \varsigma)^{\upsilon-\alpha-1} g^{(\upsilon)}(\varsigma) d\varsigma \quad (3.5)$$

where $\upsilon \in \aleph$ with $\upsilon - 1 < \alpha < \upsilon$.

4 Main Results

We will employ iterative processes (2.3), (2.4),(2.6), (2.7), and (2.8) to find the approximate solutions of the fractional differential equations. We consider Caputo's fractional differential equation with boundary conditions:

$${}^{c}D_{a+}^{\alpha} u(t) = \varphi(t, u(t)) \ , \ t \in \Im \quad (4.1)$$
$$u(a) = u_a \ , \ u(b) = u_b$$

where $0 < \alpha \leq 1, u(t) \in C(\Im, \Re)$ and $\left(C(\Im, \Re), \| \ \|_{\infty}\right)$ is Banach space, where $\|u - u^*\|_{\infty} = \sup_{t \in \Im} |u(t) - u^*(t)|$, consequently, φ is a continuous function from $I \times \Re$ to $\Re$, and there exists a constant $L > 0$ such that $|\varphi(t,u) - \varphi(t,u^*)| \leq L |u - u^*|$, $\forall t \in \Im, u, u^* \in C(\Im, \Re)$, L is a Lipschitzian constant.

Lemma 4.1. Assume that $\varphi : \Im \times \Re \to \Re$ is a continuous function, then the fractional differential equation of boundary value problem (4.1) has a unique solution:

$$u(t) = u(a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} \varphi(\tau, u(\tau)) d\tau, \quad (4.2)$$

in $C(\Im, \Re)$ and the sequence of successive approximations $\{u_i\}$ defined by:

$$u_{i+1}(t) = u_0(a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} \varphi(\tau, u_i(\tau)) d\tau, \ t \in \Im, i \geq 0$$

converges to u as $m \to \infty$.

Lemma 4.2. Assume $\varphi(t, u(t))$ is a continuous function on $\Im \times \Re$. Then $u(t) \in C(\Im, \Re)$ is a solution of the BVP (4.1) if and only if $u(t)$ is a solution of (4.2).

Lemma 4.3. Suppose that $\varphi(t, u(t))$ is a continuous on $\Im \times \Re$, and define the mapping F from $C(\Im, \Re)$ into itself by:

$$Fu(t) = u(a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} \varphi(\tau, u(\tau)) d\tau$$

Then F is a completely continuous.

***Proof*.** For any $u \in C(\Im, \Re)$ from definition of $Fu(t)$ and lemma (4.1), we have $Fu \in C(\Im, \Re)$ and $Fu(t) \geq 0$, $t \in \Im$. Hence $F(C(\Im, \Re)) \subset C(\Im, \Re)$. Since $\varphi(t, u(t)) \in C(\Im \times \Re)$ the continuity of F is obvious. Now, let $\Omega_{\varepsilon} \subset C(\Im, \Re)$ be bounded; where $\Omega_{\varepsilon} = \{u \in C(\Im, \Re) : \|u\|_{\infty} < \varepsilon\}$, let $M = \max_{t \in \Im} |\varphi(t, u(t))|$, then for that $u \in \Omega_{\varepsilon}$, from the lemma (4.1), we have

$$|Fu(t)| \leq |u(a)| + \frac{1}{\Gamma(\alpha)} \int_a^t |(t - \tau)^{\alpha-1}| |\varphi(\tau, u(\tau))| d\tau \leq |u_a| + \frac{M(t-a)^{\alpha}}{\Gamma(\alpha+1)} \quad (4.3)$$

Hence, $F(\Omega_{\varepsilon})$ is bounded. Let $t_1, t_2 \in \Im, t_1 < t_2$, and $u \in \Omega_{\varepsilon}$. Then:

$$|Fu(t_1) - Fu(t_2)| = \left| \frac{1}{\Gamma(\alpha)} \int_a^{t_1} (t - \tau)^{\alpha-1} \varphi(\tau, u(\tau)) d\tau - \frac{1}{\Gamma(\alpha)} \int_a^{t_2} (t - \tau)^{\alpha-1} \varphi(\tau, u(\tau)) d\tau \right|$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t - \tau)^{\alpha-1} |\varphi(\tau, u(\tau))| d\tau$$
$$\leq \frac{M}{\Gamma(\alpha+1)} [\chi_1 - \chi_2] \quad (4.4)$$

where $\chi_j = (t - t_j)^{\alpha}, j = 1, 2$, the Arzela-Ascoli theorem guarantees that $F(\Omega_{\varepsilon})$ is relatively compact, which means that F is compact. Thus, F is completely continuous.

Theorem 4.2. Assume that $\varphi(t,u(t))$ is a given and satisfied the following conditions:

(***H₁***) the function $\varphi:\Im\times\Re\to\Re$ is a continuous.
(***H₂***) There exist $0<L<1$ such that:
$\left|\varphi(t,u)-\varphi(t,u^*)\right|\le L\left|u-u^*\right|$, for any $t\in\Im$,
$u,u^*\in C(\Im,\Re)$.

(***H₃***) $L\dfrac{(t-a)^\alpha}{\Gamma(\alpha+1)}<1$.

Then eq.(4.1) has a unique solution q in $C(\Im,\Re)$, and the Picard iteration is a converging to q.

Theorem 4.3. Assume that $\varphi(t,u(t))\in\left(C(\Im\times\Re,\Re)\right)$ is a given, and satisfied condition (***H₁***)-(***H₃***). Then the problem (4.1) has a unique solution q in $C(\Im,\Re)$ and the Mann iterative process converges to q.

Proof : Let a sequence $\{u_i\}$ generated by the iteration (2.4) . Now, we define F as the form:

$$Fu(t)=u_a+\frac{1}{\Gamma(\alpha)}\int_a^t(t-\tau)^{\alpha-1}\varphi(\tau,u(\tau)d\tau,\quad t\in\Im \tag{4.5}$$

So, for each $t\in\Im, i=0,1,...$, we have:

$$\begin{aligned}\|u_{i+1}-q\|_\infty&=\|(1-\eta_i)(u_i-q)+\lambda_i(Fu_i-q)\|_\infty\\&=(1-\eta_i)\|u_i-q\|_\infty+\eta_i\sup_{t\in\Im}|Fu_i(t)-Fq(t)|\\&=(1-\eta_i)\|u_i-q\|_\infty+\frac{\eta_i}{\Gamma(\alpha)}\sup_{t\in\Im}\left|\int_a^t(t-\tau)^{\alpha-1}|\varphi(\tau,u_i(\tau))-\varphi(\tau,q(\tau))|d\tau\right|\end{aligned}$$

$$\|u_{i+1}-q\|_\infty\le(1-\eta_i)\|u_n-q\|_\infty+\frac{L\eta_i}{\Gamma(\alpha)}\int_a^t(t-\tau)^{\alpha-1}\sup_{t\in\Im}|u_n(\tau)-q(\tau)|d\tau$$

$$\le\ [(1-\eta_i)+L\eta_i\frac{(t-a)^\alpha}{\Gamma(\alpha+1)}]\|u_i-q\|_\infty. \tag{4.6}$$

Since $(1-\eta_i)+\dfrac{L\eta_i(t-a)^\alpha}{\Gamma(\alpha+1)}<1$, hence, the conditions stipulated by Lemma 2.1. have been fulfilled, and consequently, $u_i\to p$ as $i\to\infty$ according to Banach's fixed point theorem (Chipot, 2011, Granas & Dugundji, 2003), the operator F has a unique fixed point, which implies that eq.(4.1) has a unique solution q in $C(\Im,\Re)$.

Theorem 4.3. Assume that $\varphi(t,u(t))\in\left(C(\Im\times\Re,\Re)\right)$ is a given and satisfied condition (***H₁***)-(***H₃***). Then eq.(4.1) has a unique solution q in $C(\Im,\Re)$ and the iteration process (2.6) converges to q.

Proof: Let $\{v_i\}$ generated by the Picard-Mann hybrid process (2.6), and define F as:

$$Fv(t)=v(a)+\frac{1}{\Gamma(\alpha)}\int_a^t(t-\tau)^{\alpha-1}\varphi(\tau,v(\tau)d\tau,\quad t\in\Im$$

Thus, for each $t\in\Im, i=0,1,...$

$$\begin{aligned}\|y_i-q\|_\infty&=\|(1-\eta_i)(v_i-q)+\eta_i(Fv_i-q)\|_\infty\\&=(1-\eta_i)\|v_i-q\|_\infty+\eta_i\sup_{t\in\Im}|Fv_i(t)-Fq(t)|\\&=(1-\eta_i)\|v_i-q\|_\infty+\frac{\eta_i}{\Gamma(\alpha)}\sup_{t\in\Im}\left|\int_a^t(t-\tau)^{\alpha-1}|\varphi(\tau,v_i(\tau))-\varphi(\tau,q(\tau))|d\tau\right|\\&\le[(1-\eta_i)+\frac{L\eta_i}{\Gamma(\alpha+1)}(t-a)^\alpha]\|v_i-q\|_\infty.\end{aligned} \tag{4.7}$$

Using (2.6), (2.2) and (4.7), we obtain the follow:

$$\begin{aligned}\|v_{i+1}-q\|_\infty&=\|Fy_i-Fq\|_\infty=\sup_{t\in\Im}|Fy_i(t)-Fq(t)|\\&\le\rho\|y_n-q\|_\infty\\&\le\rho[(1-\eta_i)+\frac{L\eta_i}{\Gamma(\alpha+1)}(t-a)^\alpha]\|v_i-q\|_\infty\end{aligned} \tag{4.8}$$

since $(1-\eta_i)+\dfrac{L\eta_i}{\Gamma(\alpha+1)}(t-a)^\alpha<1$, $0<\rho<1$, therefore, from Lemma 2.1. we have: $v_i\to q$ as $i\to\infty$ and according to Banach's Theorem, the operator F has a unique fixed point, which is a unique solution for eq. (4.1) in $C(\Im,\Re)$.

Theorem 4.4. Assume that $\varphi\in\left(C(\Im\times\Re,\Re)\right)$ is a given and satisfied condition (***H₁***)-(***H₃***). Then problem (4.1) has a unique solution q in $C(\Im,\Re)$, and the iteration process (2.7) converging to q.

Proof: Let $\{x_i\}$ generated by the Picard-Krasnoselskii hybrid iteration (2.7) and the operator F define by:

$$Fx'(t)=x'(a)+\frac{1}{\Gamma(\alpha)}\int_a^t(t-\tau)^{\alpha-1}\varphi(\tau,x(\tau)d\tau, \tag{4.9}$$

subsequently, for each $t\in\Im, i=0,1,...$:

$$\begin{aligned}\|y_i-q\|_\infty&\le(1-\kappa)\|x'_n-q\|_\infty+\kappa\|Fx'_i-Tq\|_\infty\\&=(1-\kappa)\|x'_i-q\|_\infty+\kappa\sup_{t\in\Im}|Fx'_i(t)-Fq(t)| \qquad (4.10)\\&=(1-\kappa)\|x'_i-q\|_\infty+\frac{\kappa}{\Gamma(\alpha)}\sup_{t\in\Im}\left|\int_a^t(t-\tau)^{\alpha-1}\left[\varphi(\tau,x'_i(\tau))-\varphi(\tau,q(\tau))\right]d\tau\right|\\&\le(1-\kappa)\|x'_i-q\|_\infty+\frac{L\kappa}{\Gamma(\alpha)}\int_a^t(t-\tau)^{\alpha-1}\|x'_i-q\|_\infty d\tau\end{aligned}$$

$$\|y_i-q\|_\infty\le\ [(1-\kappa)+\frac{L\kappa}{\Gamma(\alpha+1)}(t-a)^\alpha]\|x'_i-q\|_\infty. \tag{4.11}$$

By use (2.7) , (2.2) and (4.11) we get the follow:

$$\begin{aligned}\|x'_{i+1}-q\|_\infty&=\|Fy_n-Fq\|_\infty=\sup_{t\in\Im}|Fy_i-Fq|\\&\le\rho\|y_n-q\|_\infty \qquad (4.12)\\&\le\ \rho[(1-\kappa)+\frac{L\kappa}{\Gamma(\alpha+1)}(t-a)^\alpha]\|x'_i-q\|_\infty.\end{aligned}$$

Since $(1-\kappa)+\dfrac{L\kappa}{\Gamma(\alpha+1)}(t-a)^\alpha<1$ and $\rho\in(0,1)$, then the of Lemma 2.1. are satisfied, which implies that $x'_i \to q$ as $i \to \infty$ and by applying Banach's theory, it follows that F has a unique fixed point, which means that q is a unique solution for eq.(4.1) in $C(\Im, \Re)$.

Theorem 4.5. Assume that $\varphi \in (C(\Im \times \Re, \Re))$ is a given, and satisfied condition (***H*₁**)-(***H*₃**), then eq.(4.1) has a unique solution q in $C(\Im, \Re)$ and the Ishikawa iterative process (2.8) converges to q.

Proof : Let a sequence $\{w_i\}$ generated by the Ishikawa process (2.8). Now, we define the operator F as:

$$Fw(t) = w(a) + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \varphi(\tau, \mathrm{w}(\tau)) d\tau, \quad t \in \Im \tag{4.13}$$

for each $t \in \Im, i = 0,1,...$ we have:

$$\begin{aligned} \|y_i - q\|_\infty &= \|(1-\xi_i)(w_i - p) + \xi_i (Fw_i - q)\|_\infty \\ &\le (1-\xi_i)\|w_i - q\|_\infty + \xi_i \sup_{t\in I} |Fw_i(t) - Fq(t)| \\ &\le [(1-\xi_i) + \frac{L\xi_i}{\Gamma(\alpha+1)}(t-a)^\alpha]\|w_i - q\|_\infty . \end{aligned} \tag{4.14}$$

Consequently, from (2.8), (2.2) and (4.14) we obtain the following:

$$\begin{aligned} \|w_{i+1} - q\|_\infty &= \|(1-\eta_i)w_n + \eta_i F y_i - q\|_\infty \\ &\le (1-\eta_i)\|w_i - p\|_\infty + \eta_i \rho \|y_i - q\|_\infty \\ &\le (1-\eta_i)\|w_i - p\|_\infty + \eta_i \rho[(1-\xi_i) + L\xi_i \frac{(t-a)^\alpha}{\Gamma(\alpha+1)}]\|w_i - q\|_\infty \\ &\le \left(1 - \eta_i[1 - \rho[(1-\xi_i) + \frac{L\xi_i}{\Gamma(\alpha+1)}(t-a)^\alpha]\right)\|w_i - q\|_\infty \\ &\le \left(1 - \eta_i[1 - \rho[(1-\xi_i(1 - \frac{L}{\Gamma(\alpha+1)}(t-a)^\alpha]\right)\|w_n - q\|_\infty \end{aligned} \tag{4.15}$$

since $1 - \eta_i[1 - \rho[(1-\xi_i(1 - \frac{L}{\Gamma(\alpha+1)}(t-a)^\alpha] < 1$, so $w_n \to q$, through the Banach's Theorem we get $q \in fixd(F)$, and conclude that the eq.(4.1) has a unique solution q in $C(\Im, \Re)$.

Now, we are in a position to give the following proposition for the comparison between the previous iterative processes.

Proposition 4.1 Let $M \ne \phi$ be a bounded and closed convex subset of Banach space A, the mapping F from M into itself is a contraction, assume that each processe: (2.3),(2.4),(2.6),(2.7) and (2.8) converges to the same fixed point of F, where $\{\eta_i\}_{i=0}^\infty, \{\xi_i\}_{i=0}^\infty$ are sequences in the interval $(0,1)$ such that:

$0 < \gamma \le \eta_i, \xi_i < 1$ for all $i \ge 0$ and for some $\gamma \ge 0$, $\kappa \in (0,1)$. Then the Picard-Mann hybrid iteration (2.6) converges faster than the other iterations.

Proof: Suppose that $q \in fixd(F)$, and given that F is a contraction mapping from (2.2) and Picard iteration (2.3) we have:

$$\begin{aligned} \|x_{i+1} - p\|_\infty &= \|Fx_i - q\|_\infty \\ &\le \rho \|x_i - q\|_\infty \\ &\vdots \\ &\le \rho^i \|x_1 - q\|_\infty \end{aligned} \tag{4.16}$$

Let:

$$a_i = \rho^i \|x_1 - q\|_\infty \tag{4.17}$$

So, from (2.2) and the Mann iteration (2.4),we have:

$$\begin{aligned} \|u_{i+1} - q\|_\infty &= \|(1-\eta_i)(u_i - p) + \eta_i (Fu_i - q)\|_\infty \\ &= (1-(1-\rho)\eta_i)\|u_i - q\|_\infty \\ &\le (1-(1-\rho)\gamma)\|u_i - q\|_\infty \\ &\vdots \\ &\le (1-(1-\rho)\gamma)^i \|u_1 - q\|_\infty \end{aligned} \tag{4.18}$$

Let:

$$b_i = (1-(1-\rho)\gamma)^i \|u_1 - q\|_\infty \tag{4.19}$$

By using (2.2)and (2.6), we obtain:

$$\begin{aligned} \|v_{i+1} - q\|_\infty &= \|Fy_i - q\|_\infty \\ &\le \rho \|y_i - q\|_\infty \\ &\le \rho\left[(1-\eta_i)\|v_i - q\|_\infty + \rho\eta_i \|v_i - q\|_\infty\right] \\ &= \rho(1-(1-\rho)\eta_i)\|v_i - q\|_\infty \\ &\le \rho(1-(1-\rho)\gamma)\|v_i - q\|_\infty \\ &\vdots \\ &\le \rho^i (1-(1-\rho)\gamma)^i \|v_1 - q\|_\infty \end{aligned} \tag{4.20}$$

Put:

$$c_i = \rho^i (1-(1-\rho)\gamma)^i \|v_1 - q\|_\infty \tag{4.21}$$

By using (2.2) and eq. (2.7), we obtain:

$$\begin{aligned} \|x'_{i+1} - q\|_\infty &= \|Fy_i - q\|_\infty \le \rho \|y_i - q\|_\infty \\ &\le \rho \|(1-\kappa)(x'_i - q) + \kappa(Fx'_i - q)\|_\infty \\ &= \rho(1-(1-\rho)\kappa)\|x'_i - q\|_\infty \\ &\vdots \\ &\le \rho^i (1-(1-\rho)\kappa)^i \|x'_1 - q\|_\infty \end{aligned} \tag{4.22}$$

Since $0 < \kappa < 1$, then for $i \ge 0$ we have $1-(1-\rho)\kappa < 1$

Put:

$$d_i = \rho^i \|x'_1 - q\|_\infty \tag{4.23}$$

Now, from (2.2) and the Ishikawa iteration (2.6), it follows that:

$$\begin{aligned}\|y_i - q\|_\infty &= \|(1-\xi_i)w_n + \xi_i F w_i - p\|_\infty \\ &\le (1-\xi_i)\|w_i - q\|_\infty + \xi_i \rho \|w_i - q\|_\infty \\ &\le [(1-\xi_i) + \rho\xi_i]\|w_i - q\|_\infty\end{aligned} \quad (4.24)$$

From (2.2), (2.6), and (4.24), we obtain that

$$\begin{aligned}\|w_{i+1} - q\|_\infty &= \|(1-\eta_i)(w_i - q) + \eta_i (Fy_i - q)\|_\infty \\ &\le (1-\eta_i)\|w_i - q\|_\infty + \rho\eta_i\left[(1-\xi_i) + \xi_i\rho\right]\|w_i - q\|_\infty \\ &\le \left(1-\eta_i + \rho\eta_i\left[(1-\xi_i)+\xi_i\rho\right]\right)\|w_i - q\|_\infty \\ &\le \left(1-\eta_i(1-\rho) - \rho\eta_i\xi_i(1-\rho)\right)\|w_i - q\|_\infty \\ &\vdots \\ &\le \left(1-\gamma(1-\rho) - \rho\gamma^2(1-\rho)\right)^i \|w_1 - q\|_\infty\end{aligned} \quad (4.25)$$

for $i \ge 0$. Since $0 \le \gamma \le \eta_i, \xi_i \le 1$, then for $i \ge 0$ we have: $1-\eta_i(1-\rho) - \rho\eta_i\xi_i(1-\rho) \le 1-\gamma(1-\rho) - \rho\gamma^2(1-\rho) < 1$

Put:

$$e_i = \|w_1 - p\|_\infty \quad (4.26)$$

We now compare the convergence of the Picard-Mann hybrid iteration process (2.6) as follows:

Firstly, we calculated the convergence rate between the Picard-Mann hybrid and iterations (2.3), we obtained:

$$\frac{c_i}{a_i} = \frac{(1-(1-\rho)\gamma)^i \|v_1 - q\|_\infty}{\|x_1 - q\|_\infty} \to 0 \ as \quad i \to \infty \quad (4.27)$$

Secondly, we calculate the convergence rate between the iterative process (2.6) and Mann iteration methods:

$$\frac{c_i}{b_i} = \rho^i \frac{\|v_1 - q\|_\infty}{\|u_1 - q\|_\infty} \to 0 \ as \quad i \to \infty$$

(4.28)

Thirdly, we computed the convergence rate between the iterative process (2.6) and the Picard-Krasnoselskii hybrid iteration process:

$$\frac{c_i}{d_i} = (1-(1-\rho)\gamma)^i \frac{\|v_1 - q\|_\infty}{\|x'_1 - q\|_\infty} \to 0 \ as \quad i \to \infty \quad (4.29)$$

Finally, we calculated the convergence rate between the iterative process (2.6) and the Ishikawa iteration process:

$$\frac{c_i}{e_i} = \frac{\rho^i (1-(1-\rho)\gamma)^i \|v_1 - q\|_\infty}{\|w_1 - q\|_\infty} \to 0 \ as \quad i \to \infty \quad (4.30)$$

Consequently, we conclude from (4.27), (4.28), (4.29), (4.30) that the speed convergence of $\{v_i\}_{i=1}^\infty$ to q is better than both $\{x_i\}_{i=1}^\infty, \{u_i\}_{i=1}^\infty, \{x'_i\}_{i=1}^\infty$ and $\{w_i\}_{i=1}^\infty$.

5 Illustrative Example

We introduce the following example to illustrate the previous comparison in the proposition:

Example 1. we consider Caputo Fractional differential equation for $\alpha = 1/2$ with conditions:

$$\begin{cases} {}^C D^\alpha_{a=0} u(t) = Exp(-t)u(t), & t \in [0,1] \\ u(0) = 1, \ {}^C D^\alpha u(0) = 1 \end{cases}$$

where $\kappa = 1/3$ and $\{\eta_i\}_{i=0}^\infty = \{10^{-2i}\}_{i=0}^\infty, \{\beta_i\}_{i=0}^\infty = \{1/2^{(i+1)!}\}_{i=0}^\infty$.

it's clear that: $\Phi = Exp(-t)u(t) \in \left(C([0,1]\times\Re, \Re)\right)$ was satisfied the conditions (***H*1**)-(***H*3**), further, for each $t \in [0,1]$ and $u, u^* \in \Re$, we get:

$$\left|\Phi(t,u) - \Phi(t,u^*)\right| = |e^{-t}| \left|u - u^*\right| \le \left|u - u^*\right|$$

The following tables show that the Picard-Mann hybrid iterative is better convergent than all other processes to the exact solution.

t_i	Exact	Picard	Mann	Krasnoselskii
0.	1.	1.	1.	1.
0.1	1.10517	1.33396	1.00334	1.33396
0.2	1.2214	1.44243	1.00442	1.44243
0.3	1.34986	1.50807	1.00508	1.5080
0.4	1.49182	1.5506	1.00551	1.5506
0.5	1.64872	1.57829	1.00578	1.57829
0.6	1.82212	1.59567	1.00596	1.59567
0.7	2.01375	1.60559	1.00606	1.60559
0.8	2.22554	1.60995	1.0061	1.60995
0.9	2.4596	1.61014	1.0061	1.61014
1.	2.71828	1.60716	1.00607	1.60716

Table 1

t_i	Picard–Mann	Picard- Krasnoselskii.	Ishikawa
0.	1.	1.	1.
0.1	1.45199	1.38855	1.00404
0.2	1.6859	1.56505	1.00595
0.3	1.87722	1.70443	1.00748
0.4	2.04333	1.82354	1.00878
0.5	2.19135	1.92904	1.00995
0.6	2.32523	2.02443	1.011
0.7	2.44748	2.11187	1.01196
0.8	2.55992	2.19281	1.01285
0.9	2.6639	2.26828	1.01367
1.	2.7605	2.33909	1.01444

Table 2

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