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Formulate the Matrix Continued Fractions and Some Applications

Naglaa F. A. Elfasy

Mathematics Department, Higher Institute of Sciences, Alabyar, Libya.

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A B S T R A C T

A matrix continued fraction is a matrix representation of a continued fractions, It has the following formula:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{..+\cfrac{1}{a_n}}}}$$

The matrix can be used to convert a continued fraction to a rational number by using matrix multiplication to calculate the matrix product of the continuous fraction matrix and the vector [1, 0]. Additionally, it can be used to calculate the convergent of a continued fraction by using matrix multiplication to calculate the matrix product of the continuous fraction matrix and the vector [1, 1]. It can also be used to represent and calculate the solutions of some type of recursive equations. The use of matrix representation of continued fractions allows for efficient computation of continued fraction expansions using matrix multiplication, which can be easily parallelized in parallel computation algorithms. This can lead to significant speedup in the computation of continued fractions and can be useful in various fields such as computer graphics.

1 Introduction

Continued fractions have many important properties and applications in mathematics, including in number theory, Diophantine equations, and the theory of irrational numbers[1]. They can also be used to symbolize a variety of mathematical operations, such as the logarithm, trigonometric functions, and the Riemann zeta function. In matrix form, continued fractions are used in the study of linear recurrent sequences, which are sequences of numbers that are determined by a fixed set of initial conditions and a set of recurrence relations. They have many applications in areas such as statistics, physics, and control theory[2].A matrix continued fraction is a type of representation for matrices, similar to how continued fractions represent real numbers. It is a method of approximating a matrix as the product of simpler matrices. The matrix continued fraction provides a way to decompose a given matrix into a series of simple matrices that can be easier to work with[3, 4].

Some applications of matrix continued fractions include[5, 6]:

1. Solving linear equations: it is possible to resolve systems of linear equations using matrix continuing fractions.
2. Eigenvalue approximation: to roughly determine a matrix's eigenvalues, one can use matrix continued fractions.
3. Matrix inversion: you may quickly determine a matrix's inverse by using matrix continued fractions.

Corresponding author: E-mail: almaaly506@gmail.com

4. Matrix polynomials: matrix continued fractions can be used to represent matrix polynomials, which are polynomials in which the coefficients are matrices.
5. Control systems: matrix continued fractions can be used in the design of control systems, such as linear quadratic regulator (LQR) controllers.
6. Signal processing: matrix continued fractions can be used in digital signal processing to model linear systems.
7. Orthogonal polynomials: matrix continued fractions can be used to compute orthogonal polynomials, which have important applications in numerical analysis and computational mathematics.

Continued fractions and matrices are two important mathematical concepts with an extensive variety of usage scenarios across multiple fields. Continued fractions, by means of way of expressing numbers, have many properties and applications, particularly in number theory and irrational numbers. Matrices are a way to represent and manipulate large sets of data and equations, and have many applications in linear algebra, statistics, physics, and engineering [7].

1.1 Continued Fraction Formula

A continued fraction is a representation of the form [3, 4]:

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{..+\cfrac{1}{a_3}}}}$$

a_0, a_1, a_2, and a_3 are all integers. This is a way of representing a rational number as the sum of an integer and a sequence of nested fractions.

It is also represented as [a_0, a_1, a_2, a_3,]

For example, the continued fraction representation of the number 2.5 (which is equal to 5/2) is [2, 2].

- Derive an Equation for Continued Fractions Formula

A type of expression that expresses a rational number as the combination of an integer and a series of fractional terms is called a Continued Fraction. We can derive a general equation for a continued fraction by breaking it down into its component parts.

Let us examine a type of continued fraction expressed as [5]:

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{..+\cfrac{1}{a_3}}}}$$

where the coefficients, a_0, a_1, a_2 and a_3 are integers.

$$a_0+\cfrac{b_1}{a_1+\cfrac{b_2}{\cfrac{b_3}{a_3+\cdots}}}$$

where $a_1$and $b_1$can be any complex numbers.

We can start by defining a sequence of nested fractions as:

$$f_1 = a_1$$

$$f_2 = a_2 + (\frac{1}{f_1})$$

$$f_3 = a_3 + (\frac{1}{f_2})$$

$$f_n = a_n + (\frac{1}{f_n}\text{-}1)$$

Here n is the sequence's number of terms.

Now we can substitute this sequence of nested fractions into the continued fraction equation:

$$a_0 + (1/f_1)$$

This is the general equation for a continued fraction.

Alternatively we can use the recursive relationship for the continued fraction, where:

$$a_n = p_n/q_n = p_{n+1} - q_n/q_{n+1},$$

where a_n and q_n are integers, so the equation becomes:

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{..+\cfrac{1}{a_n}}}}$$

This recursive relationship is useful for calculating the value of a continued fraction in the next step. The process is repeated until the remainder is zero and convergent.

For example, the number p_i can be represented as a continued fraction as follows:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}$$

To determine the Continued Fraction expression of a number, you can use the Euclidean algorithm to find the quotient and remainder at each step. The integer part of the quotient becomes the next term in the continued fraction, and the remainder is used as the numerator

Here is an example of how to determine the Continued Fraction expression of π through the use of the Euclidean Algorithm:

π = 3.14159265...

Start with the numerator as π and the denominator as 1.

π / 1 = 3.14159265... with a remainder of 0.14159265...

Use the integer part of the quotient as the next term in the continued fraction, and the remainder as the numerator in the next step.

3 + 0.14159265... / 1 = 3 + 0.14159265...

Repeat the process with the new numerator and denominator.

0.14159265... / 1 = 0.14159265... , With a remainder of 0.00159265...

Repeat the process until the remainder is zero.

+ 0.00159265... / 1 = 7 + 0.00159265...

The expression of π as a continued fraction is [3, 7, 15, 1, 292, ...]

Note that this is only an example of the first few steps to find the representation of π, and the exact representation can be infinite and not a finite number of terms .Other continued fraction expansions. Continued Fractions can also express other mathematical constants such as π and the square root of 2, along with certain irrational numbers. As an illustration, π has the Continued Fraction expansion [3; 7, 15, 1, 292….,], while the square root of 2 is expressed as [1; 2, 2, 2, 2, ...]. These expressions can offer precise approximations for the values of these mathematical constants.

2. Materials and Methods

2.1 Applications of Continued Fractions

Continued fractions have many applications in mathematics and other fields. Some of the most notable applications include:

- Continued fractions provide accurate approximations of irrational numbers, particularly in computer arithmetic and numerical analysis. This is particularly useful in computer arithmetic and numerical analysis.
- Continued fractions can solve specific Diophantine equations, which are equations that seek integer solutions.
- Number theory: Continued fractions have been used to prove important theorems in number theory, such as the uniqueness of continued fraction expansions for certain numbers.
- Control theory: In control theory, continued fractions represent a system's transfer function, indicating its behaviour to varied inputs.
- Cryptography: Continued fractions have been used in the design of certain cryptographic systems, such as the RSA algorithm.
- Quantum Mechanics: Continued fractions can also be used to find the energy levels of a quantum mechanical system.
- Other fields: Continued fractions are also employed in fields including signal processing, dynamical systems, statistics, and probability theory.

2.2 The Set of Continued Fractions

2.2.1 Derivation of Equation of the Set of Continued Fractions

A group of related continued fractions is called a set of continued fractions due to their shared property or connection. The specific form of the equation for a set of continued fractions will depend on the specific property or relationship being considered. For example; let's consider the set of continued fractions that represent the square roots of integers. We can derive an equation for this set of continued fractions by starting with the equation for a general continued fractions [9]:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{..+\cfrac{1}{a_3}}}}$$

If we let x be the square root of an integer, we can rewrite the equation as:

$$x=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{..+\cfrac{1}{a_3}}}}$$

We can now substitute the square root of an integer, x, into the equation:

$$\sqrt{n}=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{..+\cfrac{1}{a_3}}}}$$

where n is an integer.

Another example is the set of continued fractions that have convergent that form a Fibonacci sequence, and each convergence represents the relationship between two consecutive Fibonacci numbers. In general, the set of continued fractions can be represented by the recursive relationship for the continued fraction, where [10]:

$$a_n=p_n/q_n=p_{n+1}-q_n/q_{n+1},$$

and where pn and qn are integers, so the equation becomes:

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{..+\cfrac{1}{a_n}}}}$$

This recursive relationship is useful for calculating the value of a continued fraction and for finding its convergent.

An example of a set of continued fractions is the set of continued fractions that represent the square roots of integers. These continued fractions have the form:

$$\sqrt{n}=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{..+\cfrac{1}{a_3}}}}$$

Where n is an integer, and $a_0, a_1, a_2, a_3, \ldots$ are integers that can be determined by the method of continued fraction.

For instance, this is how √2 is represented as a continuing fraction:

√2 = [1, 2, 2, 2, ...]

This can be expressed mathematically as:

$$\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cdots}}}}}}$$

Another example is the continued fraction representation of √3:

√3 = [1, 1, 2, 1, 2, ...]

This can be expressed mathematically as:

$$\sqrt{3}=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}}$$

√3 = 1 + (1/(1 + (1/(2 + (1/(1 + (1/(2 + ...)))))))

As you can see, both examples are infinite continued fractions, which is a good indication that the square roots of integers are irrational numbers.

Another example of set of continued fractions is the set of continued fractions that have convergent that form a Fibonacci sequence, where each convergent is the ratio of two consecutive Fibonacci numbers. The continuing fraction of the golden ratio is what these are called.

φ=[1,1,1,1,1,1,...] [11]

This can be expressed mathematically as:

$$\varphi=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}}}}}$$

φ = 1 + (1/(1 + (1/(1 + (1/(1 + ...))))))

As you can see, it never ends, which is a good indication that it is an irrational number.

2.3 Continued Matrix Formula

A number is mathematically represented as the sum of an integer and a series of nested fractions using a continuing fraction formula, also known as a continued fraction. Matrixes, a different kind of mathematical object utilized in linear algebra, are not directly related to it. A rectangular array of numbers arranged in rows and columns is known as a matrix. A matrix's elements are denoted by the letter a_{ij} where i stand for the row index and j for the column index. The specific form of an equation for a matrix will depend on the operation or relationship being considered[10].

For example, the equation for matrix addition is:

$$C = A + B$$

Each element of the resulting matrix C is equal to the corresponding element of matrix A multiplied by the corresponding element of matrix B, where A, B, and C are all matrices of the same size.

The matrix multiplication equation, which is another illustration, is as follows:

C = AB,

where A and B are matrices, and C is the resulting matrix, this operation is only possible if the number of columns of the first matrix is the same as the number of rows of the second matrix. As you can see, there is no direct relationship between continued fractions and matrices; they are used in different fields of mathematics. The continued fraction expansion of a number can be represented using matrices. The matrix representation of a continued fraction is called a continued fraction matrix. The continued fraction matrix of a number x can be found by the following recursion: Let

A_0 = [1, x], and for n > 0, let A_n = [A $_{\{n-1\}}$, 0] * [0, 1; 1, (1/x)],

The continued fraction matrix of x is the matrix A_n, where n is the number of terms in the continued fraction expansion of x.

For example, let's say we want to find the continued fraction matrix of the number

x = p_i. The continued fraction expansion of p_i is [3; 7, 15, 1, 292, 1, 1, ...].

Using the recursion above, we can attain the continued fraction matrix of pi is:

A_0 = [1, p_i] A_1

= [A_0, 0] * [0, 1; 1,1/ (1/ p_i)]

= [1, p_i; 0, 3] A_2

= [A_1, 0] * [0, 1; 1, 1/ (1/(p_i - 3))]

= [1, p_i; 0, 3; 0, 7] A_3

= [A_2, 0] * [0, 1; 1, 1/(1/(p_i - 3 - 7/3))]

= [1, p_i; 0, 3; 0, 7; 0, 15]

The matrix A_n will be the matrix representation of the continued fraction of pi. It should be noted that the above representation is a general formula, and that different types of continued fractions might have different matrix representation.

The prior quotient enables us to think about matrix continuing fractions for a family of p_q matrices called Bi.

$$\Pi_n = \cfrac{1}{B_1 + \cfrac{1}{B_2 + \cfrac{1}{B_3 + \cfrac{1}{\cfrac{1}{..+B_n}}}}}$$

Definition 1: If every element of B_k is satisfied, then the continued fractions are said to be true. $B_k > 0$. A continued fraction is called periodic if there exists a positive integer k such that $B_{k+p} = B_k$ for all $p > 0$, where p is a positive integer. In other words, a continued fraction is periodic if it has a repeating pattern in its terms[13].

- bp,q(z) is a polynomial of degree sk≥1;
- bp,1(z), ..., bp, q=1(z) and b1, q(z), ..., bp=1, q(z) are polynomials of degree smaller than sk&1; all bi, j(z), i=1, ..., p=1; j=1, ..., q=1 are polynomials of degree smaller than sk=2.
- All bi, j(z), i=1, ..., p-1; j=1, ..., q-1 are polynomials of degree smaller than sk-2.

Definition 2: A regular continued fraction is a true continued fraction where all sk equal 1. This continued fraction type is also known as a simple continued fraction. It is important to note that not all true continued fractions are regular continued fractions, as some may have numerator terms that are not equal to 1.

For a regular continued fraction, the $B_k(z)$ are of the following form, with $\delta, \gamma, \beta, \alpha$ constants, and nonzero [13]:

$$B_k\,(z)=\begin{pmatrix} 0 & 0 & \gamma_{k,1} \\ 0.. & 0 & \gamma_{k,p=1} \\ \delta_{k,1} & \delta_{k,q=1} & \alpha_k z + B_k \end{pmatrix}$$

We will prove the considered functions always yield a true continued fraction, assumed to be regular.

2.4 Matrix-Valued Continued Fractions

Matrix-valued continued fractions (MVCF) are a generalization of matrix continued fractions, in which the matrices A_i in the continued fraction are not just scalar values, but are themselves matrix-valued functions. These functions can be defined on a domain, such as the complex plane, and the MVCF represents the matrix-valued function as a ratio of matrix-valued polynomials. MVCF have applications in the analysis of matrix-valued functions, such as in the study of matrix functions and operator theory. They can be used to represent solutions of certain matrix-valued differential

equations, and to approximate matrix functions in a numerically stable way.
The equation of a MVCF is similar to the one of matrix continued fraction, but the matrices A_i are matrix-valued function.

$A(z) = A_0 + [A_1(z), A_2(z), ..., A_n(z)]$

Where A(z) is the matrix-valued function being represented by the MVCF, A_0 is a matrix and
$[A_1(z), A_2(z), ..., A_n(z)]$

Is the matrix-valued fraction, also known as a matrix-valued continued fraction?
The matrix-valued fraction is defined recursively as:

$[A_1(z), A_2(z), ..., A_n(z)] = A_1(z) + A_2(z) * [A_3(z), A_4(z), ..., A_n(z)]$

Where $A_1(z)$ is a matrix-valued function, A_2(z) is a matrix-valued function and $[A_3(z), A_4(z), ..., A_n(z)]$ is a matrix-valued fraction.

2.5 The Algorithm

There are several algorithms that can be used to convert a real number into its continued fraction representation. One of the most common algorithms is the Euclidean algorithm.
The Euclidean algorithm for continued fractions proceeds as follows:

1. Start with a real number x.
2. Take the integer part of x, and call it a_0.
3. Subtract a0 from x to get the fractional part, and call it x1.
4. Invert x1 to get $1/x_1$, and call it x_2.
5. Take the integer part of x2, and call it a1.
6. Subtract a1 from x2 to get the fractional part, and call it x3.
7. Repeat steps 4-6 until a desired level of accuracy is reached.
8. The desired continued fraction representation of x is $a_0 + 1/(a_1 + 1/(a_2 + 1/(a_3 + ...)))$
9. Alternate algorithm is the Stern-Brocot tree which is a binary tree in which the vertices are the continued fractions.
10. The algorithm for generating Stern-Brocot tree proceeds as follows:
11. Start with the two fractions 0/1 and 1/0.
12. At each step, two fractions are added to the tree, one where the numerator and denominator of the left fraction are added together. , and another that is the sum of the numerator and denominator of the right fraction.

Repeat step 2 until a desired level of accuracy is reached.
The Stern-Brocot tree algorithm can be used to determine the best rational approximation of an irrational number as well as the continuing fraction representation of any real number.
In summary, there are different algorithms that can be used to convert

2.5.1 Matrix-Valued Continued Fractions Algorithm

There are different algorithms to compute matrix-valued continued fractions. One of the most commonly used algorithms is the modified Lentz's algorithm. The basic idea of the algorithm is to recursively compute the matrix-valued fraction using the following steps:

1. Start with an initial approximation of the matrix-valued function, A_0.
2. For i = 1 to n:

- Compute the matrix-valued function $B_i(z) = A_i(z)A_{\{i-1\}}(z)$
- Compute the matrix $C_i = B_i^{-1}(z)$
- Compute the matrix-valued function $D_i (z) = C_i A_i(z)$
- Update the approximation of the matrix-valued function $A_{[12]}(z) = D_i(z) + C_i A_0$
- The final approximation of the matrix-valued function is $A_n(z)$

This algorithm is widely used as it has fast convergence properties and requires only matrix-vector multiplications and inversions of matrix-valued function.
It's worth noting that this algorithm has the assumption that the matrix-valued function $A_i(z)$ is invertible for all z in the domain, also the matrix-valued functions $A_i(z)$ should have the same dimensions.
Also, it is important to note that the above algorithm is a modified version of Lentz's algorithm that is suitable for matrix-valued functions. The original Lentz's algorithm was designed for scalar-valued functions.

2.5.2 Example Matrix-Valued Continued Fractions Algorithm [15]

Here is an example of how to use the modified Lentz's algorithm to compute a matrix-valued continued fraction for a matrix-valued function A(z):
Let's assume that A(z) is a 2x2 matrix-valued function defined on the complex plane and we want to approximate it using a matrix-valued continued fraction of order n = 4.

1. Start with an initial approximation $A_0 = I$, where I is the 2x2 identity matrix.
2. For i = 1 to 4:

- Compute the matrix-valued function $B_i(z) = A_i(z)A_{\{i-1\}}(z)$, where $A_i(z)$ is given.
- Compute the matrix $C_i = B_i^{-1}(z)$, where $B_i(z)$ is given.

- Compute the matrix-valued function $D_i(z) = C_i A_i(z)$
- Update the approximation of the matrix-valued function $A_{[12]}(z) = D_i(z) + C_i A_0$

The final approximation of the matrix-valued function is $A_4(z)$

It's important to note that the above example is a simplified one and in practice, the matrix-valued function A(z) and the matrices $A_i(z)$ are typically defined in terms of more complicated expressions and operations. Also, the values of z for which we want to approximate the matrix-valued function A(z) have to be chosen carefully, taking into account the convergence properties of the algorithm.

Also, it's good to mention that, computing the inverse of a matrix-valued function can be computationally expensive and some approximations methods exist to overcome this problem.

2.6 The Properties of the Continued Fraction Matrix

A square matrix whose entries represent the coefficients of a continuing fraction expansion is known as a continued fraction matrix. These matrices have several properties, including:

1. They are triangular: The entries above the main diagonal are all zero.
2. They are invertible: Any continued fraction matrix can be inverted, and the inverse is also a continued fraction matrix.
3. They satisfy the matrix equation: $A = T_{n-1} A_0$, where T_n is the continued fraction matrix and A_0 is the initial matrix.
4. They have the property of semi-groups: The product of two continued fraction matrices is also a continued fraction matrix.
5. They are closely related to the Fibonacci sequence: The Fibonacci numbers can be calculated using a continued fraction matrix, and the continued fraction matrix can be calculated using the Fibonacci numbers.

Here are some properties of the continued fraction matrix:

1. Uniqueness: For a given real number, there is only one continued fraction representation.
2. Convergence:
   A real number converges to its continuing fractional representation
3. Periodicity: A continued fraction representation is periodic if and only if the number is a rational.
4. Monotonicity: The terms of the continued fractional representation of a number are monotonically decreasing in magnitude.
5. Inevitability: A number's continuing fraction expression can be inverted to get the number's decimal representation.
6. Continued fraction expansion: Using a procedure called the Euclidean algorithm; one can find the specific continued fraction approximation for each real integer.

2.6.1 A Numerical Example of the Properties of a Continued Fractions Matrix [18].

A numerical example of the properties of a continued fraction matrix can be demonstrated using the matrix
A = [a b; c d]

Let's assume that the entries of this matrix are the coefficients of the continued fraction expansion of a real number.

1. Triangular property:

$$A = [a\ b;\ 0\ d]$$

As you can see from the matrix A, the entries above the main diagonal are all zero.

2. Inevitability property:

$$A^{-1} = [d\ {-b};\ {-c}\ a]/\ (ad-bc)$$

The inverse of matrix A is also a continued fraction matrix.

3. Matrix equation property:

$$A_1 = [a\ b;\ c\ d] * A_0$$

Where A_1 is the next matrix in the continued fraction expansion and A_0 is the initial matrix.

4. Semi-groups property:

$$A_3 = A_2 * A_1$$

The product of two continued fraction matrices is also a continued fraction matrix.

5. Fibonacci sequence property:
   The Fibonacci numbers can be calculated using the continued fraction matrix, for example;

$$[F(n+1)\ F(n)],\ [F(n)\ F(n-1)] = [F(n+1)\ F(n)] * [1\ 1;\ 1\ 0] \qquad [21]$$

The continued fraction matrix can also be calculated using the Fibonacci numbers.

Note that this is just an example; the entries of the matrix may vary depending on the continued fraction expansion you are working with.

Theorem 1: Every true continued fraction converges to some matrix [22].

Not all genuine continuing fractions reach matrices. Real numbers can be presented as an endless or finite sequence of rational numbers using continued fractions. While some real numbers can be defined by infinitely long continued fractions that do not converge to a single value, other real numbers can be represented by finite continued fractions.

However, certain types of continued fractions, such as regular continued fractions, do converge to a matrix. The properties of the continued fraction matrix can then be used to study the properties of the corresponding real number.

The proof typically starts by assuming that the continued fraction expansion has the form:

$$b_0 + 1 / (b_1 + 1 / (b_2 + 1 / (b_3 + ...))),$$

where b0, b1, b2, b3, are the coefficients in the continued fraction expansion.

The corresponding matrix for this continued fraction is defined as:

$$A = [b_0\ b_1;\ 1\ b_2],$$

$$T = [1\ b_0;\ 0\ 1],$$

$$T^2 = [1\ b_0;\ 0\ 1] * [1\ b_0;\ 0\ 1]$$

$$= [1\ b0 + 1;\ b0\ 1],$$

$$T3 = [1\ b_0 + 1;\ b_0\ 1] * [1\ b_0;\ 0\ 1]$$

$$= [1\ b_0 + b_1 + 1;\ b_0 + 1\ b_2], ...$$

By repeatedly applying the matrix equation, it can be shown that the matrix T^n converges to a specific matrix, as n approaches infinity. The value of the continuing fraction can sometimes be determined using this matrix, also referred to as the limiting matrix. The demonstration is successful because this matrix is distinct for just about any given continuing fraction. It's worth noting that for true continued fractions, the b_n are positive integers, this is the key to the matrix converges.

Theorem 2: Assuming that the continuation from (F) is a regular multi-index of size p, where $\bar{n}=(n_1,....,n_2)$,

$$(FQ_K - P_k) = O(1/z^{n_{i+1}}), \qquad i = 1, ..., p$$
$$(FQ_K - P_k) = O(1/z^{\overline{n+1}}), \qquad i = 1, ..., p$$

Proof: The index describes aslo the regular index of size q ,$\bar{m} = (m_1 - m_2)$, and because Q_K is expanded in the basis h_0 , ..., h_k , the amount i of Q_K is of degree at most m_i for i=1, ..., q.

We have

$$(FQ_K - P_k) = (F - \Pi_k)Q_K$$

Using theorem

$$(F - \Pi\_k)_{i,j} = O(1/z^{n_i^k + m_j^k + 1})$$

With respect to z, it follows that, for i=1, ..., p,
$(FQ_k - P_k(z))t = \sum_{j=1}^{q} O(\frac{1}{z^{n_i+m_j+1}})z^{m_j} = O(\frac{1}{z^{n_i+1}})$

And from this, the weak approximation's conclusion is found.

Since this, the approximation of F is either Π_k or the two matricesQ_k and P_k,

With $\Pi_k = P_k\,(Q_k)^{-1}$ satisfying

$$(F - \Pi_k)_{i,j} = O\left(\frac{1}{z^{n_i^k + m_j^k + 1}}\right), i = 1, ..., p, \quad j = 1, ..., q$$

$$FQ_k - P_k = O(1/z^{\overline{n+1}}),$$

a matrix with entries of type 1z* on the right-hand side of the second formula, where powers of 1/z are standard multi-indices on each row and column, reducing in the rows and rising in the columns, starting from $\bar{n}$indicated by k in the first column, i.e., writing only the power and authority of the matrix,

$$O\left(\frac{1}{z^{\overline{n+1}}}\right) \text{we get , if } k = vp + \mu, o \le \mu < p$$

$$\begin{pmatrix} v+1 & ... & v+1 & v+2 & ... \\ v+1 & ... & v+1 & v+2 & ... \\ . & ... & . & .. & .. \\ v & ... & v+1 & .. & .. \\ v & v & v+1 & .. & .. \end{pmatrix}.$$

A matrix Pade approximant of F is created as a result of the continuing fraction, and because P_k is a vector polynomial, it is necessarily the polynomial part of FQ_k.

i=1,….,p; j =1,….,q,

$$f_{i,j} = \sum_{v=0}^{\infty} \frac{f_{i,j}^v}{z^{v+1}} , \Theta_{i,j}(x^v) = f_{i,j}^v$$

it respects, every functional acting on x, k defining (m_1 , ..., m_q) and $(P_k)_i$ be there the ith component of P_k

i=1,…,p , $(P_k)_i(z) = \sum_{j=1}^{q} = \Theta_{i,j}(\frac{(Q_K)_i(x)-(Q_k)_j(z)}{x-z}$

$P_k(z) = \sum_{j=1}^{q} = \Theta(\frac{(Q_K)_i(x)-(Q_k)_j(z)}{x-z})$, deg $(P_k)_i = m_i - 1$.

From this formula or from P_k=P[FQK], the degree of the components $(P_k)_i$, i=1, ..., p, is recognized: each $(Q_k)_j$ is of degree m_j for j=1, ..., q, so $(P_k)_i$ is the summation of polynomials of degree correspondingly mj&1, and so is of degree lower than or equivalent to m_1 &1 for all i among 1 and p.

Theorem 3: If and only if F is a weakly perfect matrix, the continuing fraction inferred from F is regular [24].

A weakly perfect matrix continued fraction is a specific type of matrix continued fraction that follows a specific pattern. A weakly perfect matrix continued fraction able to be signified as:

$F = [A_0, A_1, A_2, ..., A_n]$

where A_0 is a scalar or matrix, A_1, A_2, ..., A_n are matrices, and A_i^{-1} exists for all i > 0.
A regular matrix continued fraction is a matrix continued fraction that follows a specific pattern that is determined by the properties of the matrices A_i. A weakly perfect matrix continued fraction is regular because it follows a specific pattern that is determined by the properties of the matrices A_i. It follows that the continued fraction derived from F must be a weakly flawless matrix continued fraction in order for it to be true that now the continued fraction is regular.

An example of a weakly perfect matrix continued fraction is the following:

$F = [A_0, A_1, A_2, A_3]$,where;

$A_0 = [1, 2]$, A_1

$= [3, 4]$, A_2

$= [5, 6]$, A_3

$= [7, 8]$

In this example, A_0 is a 2x2 matrix and A_1, A_2, A_3 are also 2x2 matrices. The inverse of A_1, A_2, A_3 exist, so the continued fraction is weakly perfect.
Since F is a weakly perfect matrix continued fraction, its continued fraction will be regular. It can be represented as:

$F = [A_0; A_1, A_2, A_3]$

$=A_0 + 1/(A_1 + 1/(A_2 + 1/A_3))$

This continued fraction can be used to represent solutions to certain types of matrix

Here is an example of a numerical matrix continued fraction that is both weakly perfect and regular:

$F = [A_0, A_1, A_2, A_3]$

where $A_0 = [1, 2; 3, 4]$, A_1

$= [5, 6; 7, 8]$, A_2

$= [9, 10; 11, 12]$, A_3

$= [13, 14; 15, 16]$

In this example, A_0, A_1, A_2, A_3 are all 2x2 matrices and inverse of all matrices exist, it is a weakly perfect matrix continued fraction.
Thus the continued fraction is regular and can be represented as:
$F = [A_0; A_1, A_2, A_3]$
$= A_0 + 1/(A_1 + 1/(A_2 + 1/A_3))$

The continuing fraction derived by F remains hence steady if and only if F is a weakly ideal matrix. The same is true in this instance.

2.7 Applications for Matrix Continued Fractions (MCF)

Matrix continued fractions (MCF) have a varied range of applications in several fields, such as control theory, signal processing, and computer science. Some of the most notable applications include:

1. Linear systems control: MCF can be used to represent systems of linear differential equations, which are commonly used in control systems. The MCF representation can be used to design controllers for linear systems, and to analyze the stability and performance of the system.
2. Signal processing: In the frequency domain, MCF may be used to represent signals and systems. This form can be used to create digital signal filters and examine the frequency response of a system.
3. Computer science: MCF can be used to represent systems of polynomials, which are commonly used in computer science. This representation can be used to analyze the properties of polynomial systems, and to design algorithms for solving polynomial equations.
4. Robotics: MCF can be used to represent systems of linear differential equations, which are commonly used in robotics. The MCF representation can be used to analyze the dynamics of robotic systems, and to design controllers for robotic systems.
5. Linear Algebra: MCF can be used to represent the matrix and its inverse; this representation can be used to explain linear systems and inversion of matrices.
6. Optimization: MCF can be used in optimization, MCF can be used to resolve linear least squares problem, and to find the maximum of a linear function subject to linear constraints.

It's worth noting that the application of MCF is not limited to these examples and it can be used in many other areas.

2.7.1 Solution of Matrix Equations by Branching Continued Fraction

The equation to solve matrix equations by branching the continuous fraction is called the "Matrix Fractional Description (MFD)". The general form of the MFD equation is [26]:

$X = (A - BK)^{(-1)} (C + DK)$

Where X is the unknown matrix, A, B, C, D, and K are known matrices and $^{(-1)}$ denotes the matrix inverse.This equation canister be used to find the optimal fundamental of the matrix equation by using the continuous fraction expansion. The optimal value of the matrix K can be found by iteratively solving the equation and updating the values of K.

Here is a simple example to illustrate the use of the Matrix Fractional Description (MFD) equation to solve a matrix equation:

Given the following matrix equation:

$X = (A - BK)^{(-1)} (C + DK)$

Where:

A = [[2, 0], [0, 3]]

B = [[1, 2], [3, 4]]

C = [[1, 0], [0, 1]]

D = [[1, 2], [3, 4]]

K = [[k_1, k_2], [k_3, k_4]] (unknown matrix)

To find the solution for X, we need to find the optimal value of the unknown matrix K. We can do this by using the MFD equation and iteratively updating the values of K until a satisfactory solution is found.

Let's start with an initial guess for the values of K:

K = [[0, 0], [0, 0]]

Using the MFD equation, we can calculate the first iteration of X

:

$X = (A - BK)^{(-1)} (C + DK)$

=A - [[0, 0], [0, 0]])^$^{(-1)}$ (C + [[1, 2], [3, 4]])

=A)$^{(-1)}$ (C + D)

= [[0.5,0],[0,1/3]],[[1,2],[3,4]

= [[2.5,5],[9,4]]

We can now use this updated value of X to update the values of K and repeat the process until the solution converges.This is just a simple example to show the basic idea of how to solve matrix equations using the MFD equation. In practice, the process can be more complex and multiple iterations may be required to achieve an optimal solution.

2.7.2 Matrix Representation of Continued Fraction and its Use in Parallel Computation Algorithms

Deduce the matrix representation equation for the continuous fraction and use it in parallel arithmetic algorithms. The matrix representation of a continuous fraction can be represented using the RATIONMATRIX formula, which is a 2x2 matrix. The equation for RATIONMATRIX is given by:

RATIONMATRIX (a, b, c, d) = | a b | | c d |

This matrix can be used in parallel arithmetic algorithms for fast computation of continued fractions. For example, the matrix representation can be used to estimate the value of a continued fraction in parallel, which can reduce the computational time compared to traditional sequential algorithms. In parallel arithmetic algorithms, the matrix representation of a continued fraction is multiplied with a vector of intermediate values to compute the final result in parallel. The intermediate values are then combined to get the final result. This approach is more efficient than traditional sequential algorithms as the computation can be done in parallel, reducing the overall time taken for computation.

Example for the matrix representation equation for the continuous fraction and use it in parallel arithmetic algorithms

Consider the continued fraction representation of a number as follows:

$a_0 + 1/(a_1 + 1/(a_2 + 1/(a_3 + ...)))$

The matrix representation of this continued fraction can be given as:

R_0 = RATIONMATRIX(a0, 1, 0, 1)
R_1 = RATIONMATRIX(a1, 1, 0, 1)
R_2 = RATIONMATRIX(a2, 1, 0, 1) ...

The final matrix representation of the continued fraction can be calculated as the product of these matrices:

$R = R_0 * R_1 * R_2 * ...$

This final matrix R can be used in parallel arithmetic algorithms to subtract the value of the continued fraction.

For example, a parallel algorithm can be implemented as follows:

1. Initialize a vector $v_0 = [1, a_0]$
2. Divide the intermediate matrices R_0, R_1, R_2, . into equal parts and assign each part to a different processing unit.
3. Each processing unit multiplies its assigned part of the intermediate matrices with the vector v0 to get intermediate vectors v1, v2, ...
4. The intermediate vectors are combined to get the final result v = [x, y], everyplace x/y is the value of the continued fraction.

This parallel algorithm can significantly reduce the computational time compared to traditional sequential algorithms.

Consider a continued fraction representation of the number as:

$a_0 + 1/(a_1 + 1/(a_2 + 1/(a_3 + 1/(a_4))))$
$= a_0 + 1/(a_1 + 1/(a_2 + 1/(a_3 + 1/a_4)))$

The corresponding matrices for each term in the continued fraction can be given as:

R_0 = RATIONMATRIX(a_0, 1, 0, 1)
R_1 = RATIONMATRIX(a_1, 1, 0, 1)
R_2 = RATIONMATRIX(a_2, 1, 0, 1)
R_3 = RATIONMATRIX(a_3, 1, 0, 1)
R_4 = RATIONMATRIX($_{a4,}$ 1, 0, 1)

The intersection of these matrices yields the final matrix representation of something like the continuing fraction:
$R = R_0 * R_1 * R_2 * R_3 * R_4$
Let's say we have 4 processing units. The intermediate matrices R_0, R_1, R_2, R_3, R4 can be divided into 4 parts and assigned to each processing unit as follows:
Processing Unit 1: $R_0 * R_1 * R_2$
Processing Unit 2: R_3
Processing Unit 3: R_4
The intermediate vectors can be calculated as:
Processing Unit 1: $v_0 * (R_0 * R_1 * R_2) = v_0 * R_0 * R_1 * R_2$
Processing Unit 2: $v_0 * R_3$
Processing Unit 3: $v_0 * R_4$
Finally, the intermediate vectors can be combined to get the final result v = [x, y], and x/y is the continued fraction's value.
This example demonstrates how the matrix representation of a continued fraction can be used in parallel arithmetic algorithms for fast computation.

3. Results

A continuous fraction (CF) is a representation of a real number as an infinite sum of terms. A continuous fraction matrix is a matrix representation of a CF, where each element of the matrix is a fraction.
To formulate the CF matrix, first we complete a form with the CF.:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{..+\cfrac{1}{a_3}}}}$$

Where a_0, a_1, a_2, a_3, ... are integers. The CF matrix is then formed by writing each fraction as a 2x2 matrix, with the numerator being the first element and the denominator being the second element.
Example: Consider the CF for the golden ratio (1 + √5)/2.
The CF for the golden ratio is:
1 + 1/(1 + 1/(1 + ...))
The CF matrix is then:
| 1 1 | | 1 0 |
This CF matrix can be used in different ways, such as:

1. Matrix exponentiation: The nth term of the CF can be calculated by raising the CF matrix to the power of n.
2. Fibonacci numbers: The nth Fibonacci number can be calculated using the CF matrix and the Fibonacci primary values [F (0)=0, F(1)=1].
3. Converging to the golden ratio: The CF matrix can be used to approximate the golden ratio by repeatedly multiplying it with an initial vector. The result will converge to the golden ratio.

These are just a few examples of how the CF matrix can be used

4. Discussion

The use of continued fractions and their matrix representation allows for efficient computation of continued fraction expansions, which can be useful in a variety of applications. One important application is in the field of number theory, where continued fractions are used to find the best rational approximations of real numbers. This is important in many areas such as computer graphics, where approximating real numbers with rational numbers can improve the precision and accuracy of computations. Another application of continued fractions is in solving certain differential equations, however, in some circumstances, continuing fractions can be employed to achieve perfect answers.

5. Conclusions

Continued fractions have applications in cryptography, where they can be used for key generation and encryption/decryption .The use of matrix representation of continued fractions allows for efficient computation of continued fraction expansions using matrix multiplication, which can be easily parallelized in parallel computation algorithms.
This can lead to significant speedup in the computation of continued fractions and can be useful in applications such as computer graphics, cryptography, and scientific computing. To conclude, the use of continued fractions and their matrix representation allows for efficient and precise computation of real numbers, the continued fraction matrix can have practical uses across diverse fields.

Conflict of Interest: The authors declare that there are no conflicts of interest.

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