المجلة العلمية لكلية التربية / جامعة سرت / المجلد (4) العدد (2)
Faculty of Education journal / Sirte University (Vol (4) No (2))
Doi (https://doi.org/10.37375/foej.v4i2.3454)
ISSN : 3740-2791

Reordering method for solving linear system

Joud M. Abdelaziz   Souad A. Abumaryam
Sirte University – Faculty of Science - Department of Mathematics
joudmakhzoj@gmail.com   s.ahmed@su.edu.ly

Abstract

This paper introduces a method for solving systems of linear equations, applicable to both non-homogeneous and homogeneous scenarios. The method involves converting the system into homogeneous equations by incorporating the constant terms into the variables. We employ matrix analysis to classify the solutions, distinguishing between unique solutions, infinite solutions, and cases with no solutions. This approach is particularly effective for linear systems where the number of equations matches the number of unknowns, utilizing determinants as a key analytical tool. We provide illustrative examples for each scenario, along with a general solution for cases exhibiting infinite solutions.

Keywords
Linear System, Determinant, General solution.

طريقة لحل أنظمة المعادلات الخطية غير المتجانسة والمتجانسة

جود محمد عبدالعزيز   سعاد أحمد ابومريم

قسم الرياضيات – كلية العلوم – جامعة سرت

الملخّص

في هذه الورقة، نقدم طريقة لحل أنظمة المعادلات الخطية غير المتجانسة والمتجانسة. تتضمن هذه الطريقة تحويل النظام إلى معادلات متجانسة من خلال دمج الحد الثابت من كل المعادلات في المتغيرات .تعمل المصفوفة كمعيار لتحديد ما إذا كان للنظام حل فريد، عدد غير محدود من الحلول، أو عدم وجود أي حل على الإطلاق. نقدم أمثلة لكل من السيناريوهات الثلاثة الممكنة ونقدم حلاً عاماً للحالات التي تحتوي على حلول غير محدودة. بالإضافة إلى ذلك، نقدم شرحًا مفصلاً للطريقة في السياق العام، مع مثال محدد حيث يكون لأحد المتغيرات قيمة فريدة بينما تمتلك المتغيرات الأخرى قيمًا غير محدودة.

الكلمات المفتاحية
النظام الخطي، المحدد، الحل العام.

المجلة العلمية لكلية التربية / جامعة سرت / المجلد (4) العدد (2)
Faculty of Education journal / Sirte University (Vol (4) No (2))
Doi (https://doi.org/10.37375/foej.v4i2.3454)
ISSN : 3740-2791

1. Introduction

Linear algebra is a fundamental area of mathematics. One of the core components of linear algebra is the study of linear systems, which are collections of linear equations that can be solved simultaneously. A system of equations consists of several equations involving a finite number of variables, which can be categorized into two types: homogeneous and non-homogeneous.

The study of systems of linear equations dates back to ancient Babylon around 1800 BC. During this period, Babylonian mathematicians devised systematic methods for solving these equations, employing geometric interpretations and algorithms. In the 1550s, the Italian mathematician Gerolamo Cardano (1501-1576) made a significant contribution to algebra by formulating a systematic approach to solve two linear equations with two unknowns. This innovative approach not only simplified the process but also paved the way for subsequent developments in linear algebra. By the 1750s, Joseph-Louis Lagrange (1736–1813) a prominent Italian-French mathematician and astronomer whose contributions significantly advanced the field of mathematics by employing matrices in his studies of optimization problems involving real-valued functions. In the 18th century, Lagrange recognized that matrices could simplify the representation and manipulation of systems of equations, enabling more efficient solutions to complex problems. Consequently, Carl Friedrich Gauss (1777–1855), a German mathematician and astronomer, was exploring the concept of determinants. His work on this topic proved essential for understanding linear transformations and the properties of matrices. Gauss's contributions during this period were instrumental in establishing the foundational principles of linear algebra, work established the foundation for studying eigenvalues and eigenvectors, which are crucial for understanding linear transformations and have important applications in fields like stability analysis and quantum mechanics. During the 19th century, the concept of vectors became a fundamental component of linear algebra. Sir William Rowan Hamilton (1805–1865), an Irish mathematician, astronomer, and physicist, and Hermann Grassmann (1809–1877), a German mathematician, played crucial roles in the formal development of vector spaces. Their contributions facilitated a more geometric interpretation of linear algebraic concepts.

These advancements deepened the understanding of linear transformations and significantly enhanced the ability to solve complex problems in fields like physics and engineering, paving the way for further innovations in mathematics and its applications. The study of linear systems remains a cornerstone of linear algebra, with foundational methods for solving linear paving the way for its evolution into a more formalized discipline. The term "system of linear equations" was formalized in the 17th century by the French mathematician, philosopher, and scientist René Descartes (1596–1650). His work linked algebra and geometry through Cartesian geometry in 1637 laid the groundwork for representing linear equations graphically. This integration not only advanced mathematical theory but also influenced various scientific fields, paving the way for further developments in linear algebra and its applications in areas such as engineering and physics (Aydin, 2017).

Linear algebra experienced significant advancements, largely due to the contributions of French mathematician Augustin-Louis Cauchy (1789–1857). He introduced several key concepts that would shape modern linear algebra. Cauchy developed important theorems related to determinants, which are crucial for understanding matrix properties and solving systems of linear equations. His pioneering the advantages of Gaussian elimination and Gauss-Jordan methods in both symbolic and numerical contexts.

Mary et al. (2017) reviewed three direct methods for solving systems, noting that no single method is optimal for every situation. The choice of method often depends on speed and accuracy, which are critical for effectively handling large systems due to extensive computations. Key foundational methods in linear algebra include Gaussian elimination (Strang, 2016), Cramer's Rule (Anton & Rorres, 2014), and matrix inversion (Lay et al., 2016). While Gaussian elimination is versatile for large systems, Cramer's Rule becomes impractical for $n > 3$ due to its factorial growth in determinant calculations.

Recent advancements have highlighted iterative methods, such as Jacobi and Gauss-Seidel, for solving large systems, along with enhancements in numerical stability for LU decomposition (Saad, 2003) and (Trefethen & Bau, 1997). However, these methods often prioritize computational efficiency over a equations developed by mathematicians. Lanczos (1952) developed a simple algorithm that effectively solves large systems by using convergent approximations, enhancing the methods available for tackling linear equations in a range of applications. Classical techniques for solving linear equations include Gaussian elimination (Strang, 2016), Cramer's Rule (Anton & Rorres, 2014), substitutions and matrix inversion (Lay et al., 2016). These methods utilize systematic elimination, determinant ratios, and direct matrix inversion. While Gaussian elimination is versatile and scalable for larger systems, Cramer's Rule, although elegant for small cases, becomes impractical for $n > 3$ due to its factorial growth in determinant calculations and Graphing methods can be effective for smaller systems, but they encounter significant limitations as the complexity of the system increases. Khan et al. (2015) explored matrix theory and direct methods for linear equations, comparing elimination techniques, particularly assessing

Doi (https://doi.org/10.37375/foej.v4i2.3454)
ISSN : 3740-2791

unknowns. The reordering method presented here connects non-homogeneous and homogeneous systems through a determinant-driven analysis. Unlike Cramer's Rule, which requires a non-zero determinant for identifying unique solutions, this method employs the condition $|A|=0$ to classify solutions, akin to eigenvalue problems in homogeneous systems. Understanding the nature of solutions to linear systems whether they yield a unique solution, infinitely many solutions, or none at all is essential. This classification provides valuable insights into the properties of matrices and the interrelationships between equations.

2.1.1 Reordering method for solving non-homogeneous linear equations:

Let we have three non-homogeneous and homogeneous linear equations with three unknowns:

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2 \qquad (1)$$
$$a_3x+b_3y+c_3z=d_3$$

clear theoretical understanding of solution classifications (unique, infinite, or no solution). In homogeneous systems, eigenvalue-based methods (Strang, 2016) and kernel space analysis are common, but they do not readily extend to non-homogeneous cases. While similar to parameterized solutions for infinite cases (Hoffman & Kunze, 1971), the reordering method uniquely incorporates constants into variables to transform the systems, a novel approach not previously documented. Its closest relative is the augmented matrix technique used in Gaussian elimination, but it prioritizes theoretical clarity over algorithmic scalability.

2. Main Result

2.1 Reordering method for solving linear equations: The term "Reordering method" originates from the initial step of rearranging the equations to achieve a specific form for a defined purpose. In the following, we provide a detailed explanation of this method using three non-homogeneous and homogeneous linear equations with three

This result is a contradiction, as the two matrices are not equal. Therefore, the probability of a solution exists only in the case where

$$|A| = 0 \Longrightarrow \begin{vmatrix} (a_1x - d_1) & b_1 & c_1 \\ (a_2x - d_2) & b_2 & c_2 \\ (a_3x - d_3) & b_3 & c_3 \end{vmatrix} = 0$$

By evaluating this determinant, we obtain a linear equation in "x" that can be solved to find its value. The result of this equation will correspond to one of the three:

1- If $x = k$, where k is number, then the linear system (1) has a unique solution.

2- If $k = 0$, then the linear system (1) has no solution.

3- If $0 = 0$, then the linear system (1) has infinitely many solutions, this is a consequence of the equation $(x.0 = 0)$

The next, reordering to find other variable (solving for y)

Case (2): Reordering the equations linear system (1) in the terms "y". To determine the value of the variable "y", we obtain the form

$$a_1x + (b_1y - d_1) + c_1z = 0$$
$$a_2x + (b_2y - d_2) + c_2z = 0$$

Case (1) Reorganizing the previous equations in the following manner:

$$\begin{aligned} (a_1x - d_1) + b_1y + c_1z &= 0 \\ (a_2x - d_2) + b_2y + c_2z &= 0 \\ (a_3x - d_3) + b_3y + c_3z &= 0 \end{aligned} \qquad (2)$$

This system of equations can be expressed in matrix form as follows:

$$\begin{bmatrix} (a_1x - d_1) & b_1 & c_1 \\ (a_2x - d_2) & b_2 & c_2 \\ (a_3x - d_3) & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Alternatively, it can be represented as:

$$AB = O \qquad (3)$$

Where,

$$A = \begin{bmatrix} (a_1x - d_1) & b_1 & c_1 \\ (a_2x - d_2) & b_2 & c_2 \\ (a_3x - d_3) & b_3 & c_3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ y \\ z \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Theorem.1 (Existence of Solutions): The system (3) has no solution when $|A| \neq 0$

Proof: If $|A| \neq 0$, this means that the matrix A has an inverse (A^{-1}). Multiplying both sides of the equation (4) by the inverse, we get

$$A^{-1}(AB) = A^{-1}0\,, \qquad (A^{-1}A)B = 0\,,$$

$$IB = 0, \qquad B = 0, \qquad \begin{bmatrix} 1 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This system can be expressed in matrix form as follows:

$$\begin{bmatrix} a_1 & b_1 & (c_1z-d_1) \\ a_2 & b_2 & (c_2z-d_2) \\ a_3 & b_3 & (c_3z-d_3) \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This system is likely to have a solution only if the determinant of the first matrix equals zero (as explained previously).

$$\begin{vmatrix} a_1 & b_1 & (c_1z-d_1) \\ a_2 & b_2 & (c_2z-d_2) \\ a_3 & b_3 & (c_3z-d_3) \end{vmatrix} = 0$$

By analyzing this determinant, we will derive an equation in "z" that can be easily solved, resulting in the value of "z" that satisfies the linear system (1). In this manner, we have obtained the solution.

For a generalization of the method, let we have the following linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= d_2 \qquad (4) \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n &= d_n \end{aligned}$$

Where

a_{ij} and d_i are constants for all i and j.

First: To find the value of x_1 we rewrite the linear system (4) in the following form:

$$a_3x + (b_3y - d_3) + c_3z = 0$$

This system can be expressed in matrix form as follows:

$$\begin{bmatrix} a_1 & b_1y-d_1 & c_1 \\ a_2 & b_2y-d_2 & c_2 \\ a_3 & b_3y-d_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ 1 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This system is likely to have a solution only if the determinant of the first matrix equals zero (as explained earlier).

$$\begin{vmatrix} a_1 & b_1y-d_1 & c_1 \\ a_2 & b_2y-d_2 & c_2 \\ a_3 & b_3y-d_3 & c_3 \end{vmatrix} = 0$$

By analyzing this determinant, we will derive an equation in "y" that can be easily solved. This will give us the value of "y", which satisfies the linear system (1).

Continuing this method, we analyze other rearrangements focusing on z, systematically applying the same principles to derive solutions for all variables.

Case (3): Reordering in terms "z" in the linear system (1), becomes the form

$$a_1x + b_1y + (c_1z - d_1) = 0$$
$$a_2x + b_2y + (c_2z - d_2) = 0$$
$$a_3x + b_3y + (c_3z - d_3) = 0$$

Doi (https://doi.org/10.37375/foej.v4i2.3454)
ISSN : 3740-2791

3 (0=0). In this case, the linear system (1) has infinitely many solutions, this is a consequence of the equation ($x_1.0=0$)

Secon: To find the value of x_2 in case where the system (5) has a unique solution, we will rewrite the system in the following form:

$$\begin{aligned} a_{11}x_1+(a_{12}x_2-d_1)+a_{13}x_3+\cdots+a_{1n}\,x_n&=0\\ a_{21}x_1+(a_{22}x_2-d_2)+a_{23}x_3+\cdots+a_{2n}x_n&=0 \quad (6)\\ &\vdots\\ a_{n1}x_1+(a_{n2}x_2-d_n)+a_{n3}x_3+\cdots+a_{nn}x_n&=0 \end{aligned}$$

We can express the system (6) in the following form:

$$\begin{bmatrix} a_{11} & a_{12}x_2-d_1 & a_{13} & \dots & a_{1n}\\ a_{21} & a_{22}x_2-\ d_2 & a_{23} & \dots & a_{2n}\\ & & \vdots & & \\ a_{n1} & a_{n2}x_2-d_n & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1\\ 1\\ x_3\\ x_4\\ \vdots\\ x_n \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\\ \vdots\\ 0 \end{bmatrix}$$

This system likely has a solution only under the following condition:

$$\begin{vmatrix} a_{11} & a_{12}x_2-d_1 & a_{13} & \dots & a_{1n}\\ a_{21} & a_{22}x_2-\ d_2 & a_{23} & \dots & a_{2n}\\ & & \vdots & & \\ a_{n1} & a_{n2}x_2-d_n & a_{n3} & \dots & a_{nn} \end{vmatrix} = 0$$

By analyzing the determinant, we derive a linear equation with a single unknown "x_2",which can be solved to determined its value. Similarly, for $x_3, x_4, \dots, x_n$.

$$\begin{aligned} (a_{11}x_1-d_1)+a_{12}x_2+a_{13}x_3+\cdots+a_{1n}\,x_n&=0\\ (a_{21}x_1-d_2)+a_{22}x_2+a_{23}x_3+\cdots+a_{2n}x_n&=0 \quad (5)\\ &\vdots\\ (a_{n1}x_1-d_n)+a_{n2}x_2+a_{n3}x_3+\cdots+a_{nn}x_n&=0 \end{aligned}$$

We can express the system (5) in matrix form as follows:

$$\begin{bmatrix} a_{11}x_1-d_1 & a_{12} & a_{13} & \dots & a_{1n}\\ a_{21}x_1-d_2 & a\ 22 & a_{23} & \dots & a_{2n}\\ & \vdots & & & \\ a_{n1}x_1-d_n & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1\\ x_2\\ x_3\\ \vdots\\ x_n \end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\\ \vdots\\ 0 \end{bmatrix}$$

This system has a solution only under the following condition:

$$\begin{vmatrix} a_{11}x_1-d_1 & a_{12} & a_{13} & \dots & a_{1n}\\ a_{21}x_1-d_2 & a\ 22 & a_{23} & \dots & a_{2n}\\ & \vdots & & & \\ a_{n1}x_1-d_n & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}$$

$$=0$$

By analyzing this determinant, we will derive a linear equation in a single unknown, x1x_1x1, which can be easily solved to find its value. The result of this equation will fall into one of the following three possibilities:

1- (x_1 = number). In this case, the linear system (5) has a unique 'x_1', so we have found the value of the first unknown.

2- (Number = 0). In this case, the linear system (5) does not have a solution.

Doi (https://doi.org/10.37375/foej.v4i2.3454)
ISSN : 3740-2791

Theorem 3: The system $AB = 0$ has no solution when $(|A| \neq 0)$

Proof: Since $(|A| \neq 0)$ it implies that A has an inverse (A^{-1}) . Multiplying both sides of the equation $AB = 0$ by A^{-1}, we get

$$A^{-1}(AB) = A^{-1}0$$

$$(A^{-1}A)B = 0 \Longrightarrow IB = 0 \Longrightarrow B = 0$$

This leads to:

$$\begin{bmatrix} 1 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This result is illogical, as the two matrices cannot be equal unless the system has only the trivial solution. Thus, the equation $AB = 0$ has a logical result only in the case $det(A) = 0$

$$\begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix} = 0$$

By evaluating this determinant, we will obtain a linear equation in one unknown, x, which can be solved to find its value. The result of this equation will be one of two following:

1- x=0: The linear system has only the

2.1.2 Reordering method for solving homogeneous linear equations:

Consider a homogeneous system with n equations and n unknowns:

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$a_3x + b_3y + c_3z = 0$$

We can express this system in matrix form as follows:

$$\begin{bmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This can summarize as

$$AB = 0$$

where

$$A = \begin{bmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ y \\ z \end{bmatrix}, 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Rewriting this matrix in the form $Ax = 0$, where A is the coefficient matrix, allows us to analyze the solutions via determinants.

Theorem 1: A homogeneous system has non-trivial solutions if and only if $det(A) = 0$ (Hoffman & Kunze, 1971).

Theorem 2: If $(|A| \neq 0)$ then the matrix A has an inverse.

By evaluating this determinant, we arrive at one of two possibilities:

1- $(y=0)$, then the linear system has only the trivial solution.

2- $\left(\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0\right)$, then the linear system has an infinite number of solutions.

Next, if we want to find z first, we can express the system in matrix form as follows:

$$\begin{bmatrix} a_1 & b_1 & c_1 z \\ a_2 & b_2 & c_2 z \\ a_3 & b_3 & c_3 z \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This system can have a solution only if the determinant of the first matrix equals zero (this has been explained previously).

$$\begin{vmatrix} a_1 & b_1 & c_1 z \\ a_2 & b_2 & c_2 z \\ a_3 & b_3 & c_3 z \end{vmatrix} = z \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

By evaluating this determinant, we also arrive at two possibilities:

1- $(z=0)$, then the linear system has only the trivial solution.

2- $\left(\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0\right)$, then the linear trivial solution.

2- The linear system has an infinite number of solutions.

To separate the determinant, we prefer to use the column containing the variable x for ease:

To prove the above, we have

$$\begin{vmatrix} a_1 x & b_1 & c_1 \\ a_2 x & b_2 & c_2 \\ a_3 x & b_3 & c_3 \end{vmatrix} = x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This implies that:

1- $(x=0)$leads to a trivial solution.

2- $\left(\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0\right)$ leads to an infinite number of solutions.

To find y first, we can write the system in matrix form as:

$$\begin{bmatrix} a_1 & b_1 y & c_1 \\ a_2 & b_2 y & c_2 \\ a_3 & b_3 y & c_3 \end{bmatrix} \begin{bmatrix} x \\ 1 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The determinant must again equal zero for the system to have a solution:

$$\begin{vmatrix} a_1 & b_1 y & c_1 \\ a_2 & b_2 y & c_2 \\ a_3 & b_3 y & c_3 \end{vmatrix} = y \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

ISSN : 3740-2791

only the trivial solution.

2- $(0.x_1 = 0)$, if the determinant is zero, the linear system has an infinite number of solutions.

Similarly, at use $x_3, x_4, \ \dots \ x_n$

3.Some Examples: First, let we consider some of non-homogeneous examples:

Example 1: - The following linear system

$$x + 2y - z = -1$$
$$2x - y + z = 6$$
$$x - 2y + z = 5$$

Let we have

$$\begin{vmatrix} (x+1) & 2 & -1 \\ (2x-6) & -1 & 1 \\ (x-5) & -2 & 1 \end{vmatrix} = 0$$

By analyzing this determinant using the first column, we arrive at the result.

$$x = 2$$

This result indicates that the system has a unique solution.

$$\begin{vmatrix} 1 & (2y+1) & -1 \\ 2 & (-y-6) & 1 \\ 1 & (-2y-5) & 1 \end{vmatrix} = 0$$

By analyzing this determinant using the second column, we obtain the result.

system has an infinite number of solutions.

For a generalization of the method, we consider the following linear system:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}\,x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = 0$$

Where a_{ij} are constants for all i and j .

We can express this system in matrix form as follows:

$$\begin{bmatrix} a_{11}x_1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21}x_1 & a\ _{22} & a_{23} & \dots & a_{2n} \\ & \vdots & & & \\ a_{n1}x_1 & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This system has a solution only if the determinant of the coefficient matrix equals zero:

$$\begin{vmatrix} a_{11}x_1 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21}x_1 & a\ _{22} & a_{23} & \dots & a_{2n} \\ & \vdots & & & \\ a_{n1}x_1 & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} = 0$$

By evaluating this determinant, we will obtain a linear equation in only one unknown, x_1. The outcomes will be one of the following:

1- $(x_1 = 0)$, then the linear system has

Doi (https://doi.org/10.37375/foej.v4i2.3454)
ISSN : 3740-2791

This result is a contradiction; therefore, this system does not have a solution.

Example 3: The following linear system

$$x + y - 2z = 1$$
$$2x + y + z = 3$$
$$3x + 2y - z = 4$$

By rearranging, we get

$$\begin{vmatrix} (x-1) & 1 & -2 \\ (2x-3) & 1 & 1 \\ (3x-4) & 2 & -1 \end{vmatrix} = 0$$

By evaluating this determinant using the first column, we obtain the result.

$$0 = 0$$

This result indicates that the system has an infinite number of solutions.

For general solution, we substitute $(z = k)$ into the linear system and find the values of (x, y) in terms of the variable k.

$$x + y - 2k = 1$$
$$2x + y + k = 3 \qquad (*)$$
$$3x + 2y - k = 4$$

From the two equations in the system $(*)$, we can express them as follows:

$$(x - 2k - 1) + y = 0$$
$$(2x + k - 3) + y = 0$$

$$y = -1$$

By analyzing this determinant using the third column, we arrive at the result.

$$\begin{vmatrix} 1 & 2 & (-z+1) \\ 2 & -1 & (z-6) \\ 1 & -2 & (z-5) \end{vmatrix} = 0$$

$$\Rightarrow z = 1$$

Thus, we have identified the unique solution to this system.

$$(x, y, z) = (2, -1, 1)$$

To verify whether the answer is correct, we substitute these values into the equations of the linear system.

Example 2: - The following linear system

$$x + 2y + z = 2$$
$$2x - y + z = 1$$
$$3x + y + 2z = 1$$

By reorganizing the above equations, we have

$$\begin{vmatrix} (x-2) & 2 & 1 \\ (2x-1) & -1 & 1 \\ (3x-1) & 1 & 2 \end{vmatrix} = 0$$

By analyzing this determinant using the first column, we obtain the result.

$$-6 = 0$$

$$6x + 2y - 3z = 4 \quad (**)$$
$$-4x - y + 2z = -2$$

We can express this system in the following form:

$$(2x - 2) + y - z = 0$$
$$(6x - 4) + 2y - 3z = 0$$
$$(-4x + 2) - y + 2z = 0$$

$$\begin{vmatrix} (2x-2) & 1 & -1 \\ (6x-4) & 2 & -3 \\ (-4x+2) & -1 & 2 \end{vmatrix} = 0$$

By calculating this determinant, we obtain the result.

$$0 = 0$$

This result indicates that the system has an infinite number of solutions.

For the general solution, we substitute $(z = k)$ into the linear system $(**)$ and find the values of(x, y) in terms of the variable k.

$$2x + y - k = 2$$
$$6x + 2y - 3k = 4 \qquad (***)$$
$$-4x - y + 2k = -2$$

These equations are non-homogeneous, and we will use the reordering method to solve

$$\begin{vmatrix} (x - 2k - 1) & 1 \\ (2x + k - 3) & 1 \end{vmatrix} = 0$$

By analyzing this determinant, we obtain the result.

$$x = 2 - 3k$$

From the two equations in the system $(*)$, we can rearrange them as follows:

$$x + (y - 2k - 1) = 0$$
$$2x + (y + k - 3) = 0$$

$$\begin{vmatrix} 1 & (y - 2k - 1) \\ 2 & (y + k - 3) \end{vmatrix} = 0$$

By evaluating this determinant, we arrive at the result.

$$y = 5k - 1$$

The general solution to this system is:

$$(x, y, z) = (2 - 3k, 5k - 1, k)$$

To verify the correctness of the solution, we substitute it into the equations of the linear system.

The following example is a special case:

Example 4: Let we have

$$2x + y - z = 2$$

Doi (https://doi.org/10.37375/foej.v4i2.3454)
ISSN : 3740-2791

$$(x,y,z)=\left(\frac{k}{2},2,k\right)$$

Next, we consider some sexamples of homogeneous system:

Example 5: Solve the following linear system

$$3x-y-z=0$$
$$x-2y+2z=0$$
$$x+y+z=0$$

By evaluating this determinant using the first column,

$$\begin{vmatrix}3x & -1 & -1\\ x & -2 & 2\\ x & 1 & 1\end{vmatrix}=0$$

we obtain the result. The determinant leads $x=0$, indicating only the trivial solution:

$$(x,y,z)=(0,0,0)$$

Example 6 Solve the following linear system

$$3x-2y-z=0$$
$$5x+y+z=0 \qquad (****)$$
$$2x+3y+2z=0$$

By evaluating this determinant using the first column,

them. From the two equations in the system (***), we can express them as:

$$(2x-k-2)+y=0$$

$$(6x-3k-4)+2y=0$$

$$\begin{vmatrix}(2x-k-2) & 1\\ (6x-3k-4) & 2\end{vmatrix}=0$$

By evaluating this determinant, we arrive at the result.

$$x=\frac{k}{2}$$

From the two equations in the system (***), we can express them as:

$$2x+(y-k-2)=0$$
$$6x-(2y-3k-4)=0$$

Finding the determinant yields

$$\begin{vmatrix}2 & (y-k-2)\\ 6 & (2y-3k-4)\end{vmatrix}=0$$

By analyzing this determinant, we obtain the result. $y=2$

This result indicates that the variable y has a unique solution. The general solution to this system is:

$$x = \frac{-k}{13}$$

Next, from the first two equations rewritten as:

$$3x - (2y + k) = 0$$
$$5x + (y + k) = 0$$

The determinant becomes:

$$\begin{vmatrix} 3 & -(2y+k) \\ 5 & (y+k) \end{vmatrix} = 0$$

By deciphering this determinant, we find:

$$y = \frac{-8k}{13}$$

Thus, the general solution to this system is:

$$(x, y, z) = \left(\frac{-k}{13}, \frac{-8k}{13}, k\right)$$

To verify the correctness of this solution, we substitute it back into the equations of the linear system $(****)$.

Example 7: Now, let's solve the following linear system:

$$\begin{aligned} -x + y + z &= 0 \\ 2x - y - 4z &= 0 \qquad (*****) \\ x + y - 5z &= 0 \end{aligned}$$

The determinant is:

$$\begin{vmatrix} 3x & -2 & -1 \\ 5x & 1 & 1 \\ 2x & 3 & 2 \end{vmatrix} = 0$$

we obtain the result. The determinant leads to $0.x = 0$, indicating an infinite number of solutions.

For the general solution:

We will substitute $(z = k)$ into the linear system $(****)$ to find the values of (x, y) in k.

$$3x - 2y - k = 0$$
$$5x + y + k = 0$$
$$2x + 3y + 2k = 0$$

These equations are non-homogeneous, so we will use the reordering method to solve them. From the first two equations in the system, we have:

$$(3x - k) - 2y = 0$$
$$(5x + k) + y = 0$$

The determinant is:

$$\begin{vmatrix} 3x - k & -2 \\ 5x + k & 1 \end{vmatrix} = 0$$

By evaluating this determinant, we find:

Next, from the two equations we have:

$$-x + (y + k) = 0$$

$$2x - (y + 4k) = 0$$

The new determinant is:

$$\begin{vmatrix} -1 & (y+k) \\ 2 & -(y+4k) \end{vmatrix} = 0$$

By evaluating this determinant, we find:

$$y = 2k$$

Thus, the general solution for this system is:

$$(x, y, z) = (3k, 2k, k)$$

For special Case: In some homogeneous linear systems, one of the unknowns may take a single value of zero, while the remaining unknowns have an infinite number of solutions. We illustrate this with the following example.

Example 8: Solve the following linear system

$$x + y - 2z = 0$$

$$3x - 4y + 8z = 0 \qquad (******)$$

$$2x + 3y - 6z = 0$$

The determinant is:

$$\begin{vmatrix} x & 1 & -2 \\ 3x & -4 & 8 \\ 2x & 3 & -6 \end{vmatrix} = 0$$

$$\begin{vmatrix} -x & 1 & 1 \\ 2x & -1 & -4 \\ x & 1 & -5 \end{vmatrix} = 0$$

This results in:

$$0.x = 0$$

This indicates that the system has an infinite number of solutions.

For the general solution: We substitute $(z = k)$ in the linear system $(*****)$, and find the values of (x, y) by the variable k.

$$-x + y + k = 0$$

$$2x - y - 4k = 0$$

$$x + y - 5k = 0$$

These equations are non-homogeneous, so we will apply the reordering method. From the first two equations, we rewrite them as:

$$(-x + k) + y = 0$$

$$(2x - 4k) - y = 0$$

The determinant is:

$$\begin{vmatrix} (-x+k) & 1 \\ (2x-4k) & -1 \end{vmatrix} = 0$$

By evaluating this determinant, we find:

$$x = 3k$$

Doi (https://doi.org/10.37375/foej.v4i2.3454)
ISSN : 3740-2791

$$3x - (4y - 8k) = 0$$

The new determinant is:

$$\begin{vmatrix} 1 & (y - 2k) \\ 3 & -(4y - 8k) \end{vmatrix} = 0$$

By evaluating this determinant, we find:

$$y = 2k$$

Thus, the general solution to this system is:

$$(x, y, z) = (0, 2k, k)$$

4. Conclusion

In summary, the reordering method for solving homogeneous linear systems provides a systematic approach to analyzing the existence of solutions. By expressing the system in matrix form and evaluating the determinant, we can determine whether the system has only the trivial solution or an infinite number of solutions. This method is not only concise and efficient but also versatile, applicable to both homogeneous and non-homogeneous equations. Its flexibility allows for the exploration of any variable in the system, facilitating the identification of solutions for any number of equations.

This results in:

$$0 = 0$$

This indicates that the system has an infinite number of solutions.

For the general solution, we substitute $(z = k)$ into the linear system $(******)$, and find the values (x, y) by the variable k.

$$x + y - 2k = 0$$

$$3x - 4y + 8k = 0$$

$$2x + 3y - 6k = 0$$

These equations are non-homogeneous, so we will apply the reordering method. From the first two equations, we have:

$$(x - 2k) + y = 0$$

$$(3x + 8k) - 4y = 0$$

The determinant is:

$$\begin{vmatrix} (x - 2k) & 1 \\ (3x + 8k) & -4 \end{vmatrix} = 0$$

By evaluating this determinant, we find:

$$x = 0$$

This indicates that the variable xxx has a single solution of zero. Now, from the two equations, we have:

$$x + (y - 2k) = 0$$

Doi (https://doi.org/10.37375/foej.v4i2.3454)
ISSN : 3740-2791

4. References

[1] Anton, H. (1994). Elementary linear algebra (2nd ed.). John Wiley & Sons, Inc.

[2] Aydin, S. (2017). On the history of some linear algebra concepts: From Babylon to pre-technology.

[3] Hoffman, K., & Kunze, R. (1971). Linear algebra (2nd ed.). Prentice-Hall.

[4] Saad, Y. (2003). *Iterative Methods for Sparse Linear Systems*. SIAM.

[5] Strang, G. (2016). *Introduction to Linear Algebra*. Wellesley-Cambridge Press (5th).

[6] Trefethen, L. N., & Bau, D. (1997). Numerical Linear Algebra. SIAM.

[7] Lay, D. C., Lay, S. R., & McDonald, J. J. (2016). Linear algebra and its applications. Pearson.

[8] Khan, M. A., Khan, A. A., & Khan, S. (2015). Matrix theory: A comprehensive introduction. Academic Press.

[9] Riga, D. (2016). Systems of linear equations: Theory and applications. Springer.

[10] Lanczos, C. (1952). An iteration method for the solution of the eigenvalue problem of linear differential and integral operators. Journal of Research of the National Bureau of Standards.

[11] Mary, J., Smith, A., & Johnson, R. (2017). A survey of three direct methods for solving systems of linear equations. Journal of Linear Algebra, 42(3), 215-230.

[12] Golub, G. H., & Van Loan, C. F. (2013). Matrix computations (4th ed.). Johns Hopkins University Press.