مجلة كلية التربية ـ جامعة سرت
المجلد (1) العدد (1) يناير(2022)

New Oscillation Theorems For Second Order Nonlinear Differential Equations Using The Integral Averaging Technique

AmbarkaAbdallaSalhin*, UmmulKhair Salma Din, RokiahRozita Ahmad &MohdSalmiMdNoorani

School of Mathematical Sciences, Faculty of Science and Technology, UniversitiKebangsaan Malaysia

*ridaabdo80@su.edu.ly

Abstract. By considering auxiliary functions, some new oscillation conditions are established
for the second order nonlinear forced differential equations. Examples are given to illustrate the main results.

Keywords: oscillation solutions; second order; nonlinear differential equations, integral averaging technique.

PACS:02.30.Hq

INTRODUCTION. In this paper, we are concerned with the oscillatory behavior of solutions of the second order nonlinear differential equation of the form

$(r(t)f(x'(t)))'+q(t)g(x(t))=H(t,x'(t),x(t))$ (E_1)

where $r,\ q\in C([t_0,\infty),\mathbb{R})$, $f,\ g\in C(\mathbb{R},\mathbb{R})$ and H is a continuous function on $\left[t_0,\infty\right)\times\mathbb{R}^2$.

The main objective of this paper is to contribute further in this direction and to establish sufficient conditions for the oscillatory behaviour of solutions of Eq. (E_1). As a consequence, we are able to extend and improve a number of previously known oscillation results. Some examples will be given.

Throughout this study, we restrict our attention only to the solutions of Eq. (E_1) which exist on some ray $[t_0,\infty)$, where $t_0\geq 0$ may depend on the particular solution. A regular solution is said to be oscillatory, if it has arbitrarily large zeros; otherwise it is said to be non-oscillatory. Equation (E_1) is called oscillatory if all its regular solutions are oscillatory.

Lots of work have been done on the following particular cases of eq.(E_1):

$$x''(t)+q(t)x(t)=0, \quad (E_2)$$

$$x''(t)+q(t)g(x(t))=0, \quad (E_3)$$

$$\left(r(t)x'(t)\right)'+q(t)g(x(t))=0, \quad (E_4)$$

$$\left(r(t)x'(t)\right)'+q(t)\,g(x(t))=H(t), \quad (E_5)$$

$$x''(t)+h(t)x'(t)+q(t)g(x(t))=0, \quad (E_6)$$

$$\left(r(t)x'(t)\right)'+h(t)x'(t)+q(t)g(x(t))=0, \quad (E_7)$$

$$\left(r(t)\left(x'(t)\right)^{\alpha}\right)' + q(t)g(x(t)) = H(t), (E_8)$$

$$\left(r(t)x'(t)\right)' + Q(t, x(t)) = H(t), (E_9)$$

There are a lot of paper involving the oscillation for (E_2) -(E_9), and other linear, nonlinear, damped and forced differential equations since the foundation work of Wintner [32] (see for [1-33]).

Graef et al. [10] studied the oscillatory behaviour of the solutionsof Eq. (E_1) with $f(x') = x'$ and $q(t)g(x(t)) = Q(t, x(t))$.

Yan [33]provedanother new oscillation criterion for Eq.(E_1) with $r(t) = 1, f(x') = x'$ $g(x(t)) = x(t)$ and $H(t, x(t), x'(t)) = 0.$

As a broadresearch field, the oscillation of differential equations has been widely studied by many authors (e.g., see [1-33] and the references quoted therein). The purpose of this paper is to generalize and complement some of the previous results.In particular, we intend to generalize the results of Greaf*et.al*[10] by using the integral averaging technique.

2.Main results

In this section, we will use the Riccati technique to establish sufficient conditions for oscillation of (E_1). Comparisons between our results and the previously known are presented and some examples illustrate the main results.Our first theorem regarding Eq. (E_1) is stated as follows:

Theorem1. Suppose that

$$O_1 \quad 0 < k_1 \le \frac{f(y)}{y} \le k_2 \text{ for all } y = x'(t) \ne 0; \text{ hold.}$$

Let ρ be a positive continuously differentiable function over $[T, \infty)$ such that $\rho'(t) \ge 0$ on $[T_0, \infty)$;

$$O_2 \quad \lim_{t\to\infty} \int_{T0}^{t} \frac{1}{\rho(s)r(s)} ds = \infty$$

$$Let\ R(s) = \rho(s)[q(s) - p(s)] - \frac{1}{4A}\frac{(\rho'(s))^2}{\rho(s)} r(s); A \text{ is a constant with}$$

$$O_3 \quad \int_{T0}^{\infty} R(s)ds = \infty.$$

Then all solutions of Eq. (E_1) are oscillatory.

Proof. To obtain a contradiction, suppose that $x(t)$ is a nonoscillatory solution on $[T, \infty)$, $T \ge T_0$ of Eq. (E_1). Without loss of generality, it can be supposed that $x \ne 0$. Weassume that $x(t)$ is positive on $[T, \infty)$. we use the Riccati technique putting

$$w(t)=\frac{r(t)f(x'(t))}{g(x(t))}\,, \qquad (1)$$

Differentiating (1) and using (E_1), we obtain,

$$w'(t)=\left[\frac{r(t)f(x'(t))}{g(x(t))}\right]'$$

$$=\frac{\left[r(t)f(x'(t))\right]'}{g(x(t))}-\frac{r(t)f(x'(t))g'(x(t))x'(t))}{g^2(x(t))}.$$

$$w'(t)=\frac{H(t,x'(t),x(t))}{g(x(t))}-q(t)-\frac{r(t)f(x'(t))g'(x(t))x'(t))}{g^2(x(t))}$$

Because $g'(x(t))\geq k$ and using the definition we have,

$$w'(t)\leq p(t)-q(t)-\frac{k}{r(t)}\frac{r^2(t)(f(x'(t)))^2}{g^2(x(t))}\frac{x'(t))}{f(x'(t))}$$

From O_1 and Eq. (1), we obtain

$$w'(t)\leq-[q(t)-p(t)]-\frac{k}{k_1}\frac{1}{r(t)}w^2(t)$$

Therefore,

$$\left[\frac{r(t)f(x'(t))}{g(x(t))}\right]'\leq-[q(t)-p(t)]-\frac{k}{k_1}\frac{1}{r(t)}w^2(t).$$

Multiplying by $\rho(t)$ and integrating from T to t, we obtain

$$\rho(t)\left[\frac{r(t)f(x'(t))}{g(x(t))}\right]'\leq-\rho(t)[q(t)-p(t)]-A\frac{\rho(t)}{r(t)}w^2(t);\ A=\frac{k}{k_1},$$

$$\int_T^t\rho(s)\left[\frac{r(s)f(x'(s))}{g(x(s))}\right]'ds\leq\int_T^t-\rho(s)[q(s)-p(s)]ds-\int_T^t A\frac{\rho(s)}{r(s)}w^2(s)\,ds.$$

Let $C_T=\dfrac{\rho(T)r(T)f(x'(T))}{g(x(T))}$,

by integrating by parts, we obtain

$$\frac{\rho(t)r(t)f(x'(t))}{g(x(t))}\leq C_T-\int_T^t\rho(s)[q(s)-p(s)]ds-\int_T^t A\frac{\rho(s)}{r(s)}w^2(s)\,ds$$

$$+\int_T^t\rho'(s)\frac{r(s)f(x'(s))}{g(x(s))}ds,$$

$$=C_T-\int_T^t \rho(s)[q(s)-p(s)]ds-\int_T^t A\frac{\rho(s)}{r(s)}w^2(s)\,ds+\int_T^t \rho'(s)w(s)\,ds,$$
$$=C_T-\int_T^t \rho(s)[q(s)-p(s)]ds+\int_T^t\left[-A\frac{\rho(s)}{r(s)}w^2(s)+\rho'(s)\,w(s)\right]ds. \qquad (2)$$

Because

$$-A\frac{\rho(s)}{r(s)}w^2(s)+\rho'(s)\,w(s)=\frac{-A\rho(s)}{r(s)}\left[\left(w(s)-\frac{\rho'(s)r(s)}{2A\rho(s)}\right)^2-\frac{(\rho'(s))^2r^2(s)}{4A^2\rho^2(s)}\right].$$

The inequality presented by (2) can be written as

$$\frac{\rho(t)r(t)f(x'(t))}{g(x(t))}\le C_T-\int_T^t \rho(s)[q(s)-p(s)]ds$$
$$+\int_T^t\frac{-A\rho(s)}{r(s)}\left[\left(w(s)-\frac{\rho'(s)r(s)}{2A\rho(s)}\right)^2-\frac{(\rho'(s))^2r^2(s)}{4A^2\rho^2(s)}\right]ds$$

Let $w(s)-\dfrac{\rho'(s)r(s)}{2A\rho(s)}=W(s)$

$$\frac{\rho(t)r(t)f(x'(t))}{g(x(t))}\le C_T-\int_T^t \rho(s)[q(s)-p(s)]ds-\int_T^t\frac{A\rho(s)}{r(s)}\left[W^2(s)-\left(\frac{\rho'(s)r(s)}{2A\rho(s)}\right)^2\right]ds$$

$$=C_T-\int_T^t \rho(s)[q(s)-p(s)]ds-\int_T^t\left[\frac{A\rho(s)}{r(s)}W^2(s)-\frac{\rho'^2(s)r(s)}{4A\rho(s)}\right]ds$$

$$=C_T-\int_T^t \rho(s)[q(s)-p(s)]-\frac{1}{4A}\frac{(\rho'(s))^2}{\rho(s)}r(s)ds-\int_T^t\frac{A\rho(s)}{r(s)}W^2(s)\,ds$$

$$\frac{\rho(t)r(t)f(x'(t))}{g(x(t))}\le C_T-\int_T^t R(s)\,ds \qquad (3)$$

Takingthe limit for both sides of (3) and using O_3, we find

$$\lim_{t\to\infty}\frac{\rho(t)r(t)f(x'(t))}{g(x(t))}\to-\infty \qquad (4)$$

Hence, there exists $T_1\ge T$ such that

$f(x'(t))<0,\ \forall t\ge T_1,$

$x'(t)<0,\ \forall t\ge T_1.$

Condition O_3 also implies that $\int_{T}^{\infty}\rho(s)[q(s)-p(s)]ds=\infty$, and there exists $T_2\ge T_1$ such that

$$\int_{T_1}^{T_2}\rho(s)[q(s)-p(s)]ds=0 \text{ and } \int_{T_2}^{t}\rho(s)[q(s)-p(s)]ds\ge 0,\quad \forall t\ge T_2.$$

Multiplying (E_1) by $\rho(t)$ and integrating by parts on $[T_2,t]$, we obtain

$$\rho(t)[r(t)f(x'(t))]'+\rho(t)q(t)g(x(t))=\rho(t)H(t,x'(t),x(t))$$

$$\frac{\rho(t)[r(t)f(x'(t))]'}{g(x(t))}\le-\rho(t)[q(t)-p(t)]$$

$$\rho(t)[r(t)f(x'(t))]'\le-\rho(t)g(x(t))[q(t)-p(t)]$$

$$\rho(t)[r(t)f(x'(t))]-\int_{T_2}^{t}\rho'(s)r(s)f(x'(s))\,ds\le C_{T_2}-\int_{T_2}^{t}\rho(s)g(x(s))[q(s)-p(s)]ds$$

$$\begin{aligned}\rho(t)[r(t)f(x'(t))]&\le C_{T_2}-g(x(t))\int_{T_2}^{t}\rho(s)[q(s)-p(s)]ds\\&\quad+\int_{T_2}^{t}x'(s)g'(x(s))\int_{T_2}^{s}\rho(u)[q(u)-p(u)]duds\\&\quad+\int_{T_2}^{t}\rho'(s)r(s)f(x'(s))ds\\&\le C_{T_2},\qquad \forall\ t\ge T_1,\end{aligned}$$

Where $C_{T_2}=\dfrac{\rho(T_2)r(T_2)f(x'(T_2))}{g(x(T_2))}<0.$

Therefore,

$\rho(t)r(t)f(x'(t))\le C_{T_2}.$

In view of O_1, we conclude that for $t\ge T_1$,

$$x'(t)\le\frac{C_{T_2}}{k_1}\frac{1}{r(t)\rho(t)},$$

$$x(t)\le\frac{C_{T_2}}{k_1}\int_{T_2}^{t}\frac{1}{r(s)\rho(s)}ds.$$

Finally, from O_2 $x(t)\to-\infty$ as $t\to\infty$, which contradicts assumption $x(t)$ is positive. Therefore, Eq. (E_1) is oscillatory.

Example 1. Consider the nonlinear differential equation

$$\left[\frac{1}{t}\left(13x'(t)+\frac{(x'(t))^{11}}{(x'(t))^{10}+1}\right)\right]'+\left(t+\frac{\sin t}{t}\right)x^5(t)=\frac{2x^{12}\sin t\cos(x'(t)+1)}{(x^7+1)t^3},\quad t\geq\frac{\pi}{2},$$

We note that,

$$\frac{H(t,x'(t),x(t))}{g(x(t))}=\frac{2x^{12}\sin t\cos(x'(t)+1)}{(x^7+1)t^3}\times\frac{1}{x^5(t)}\leq\frac{2}{t^3}=p(t),\quad \forall x'\in\mathbb{R},\ x\in\mathbb{R}\ \text{ and } t\geq t_0,$$

hence,

$$13<13+\frac{(x'(t))^{10}}{(x'(t))^{10}+1}<14,\ \forall y\neq 0,$$

$$R(s)=s^2+\sin s-\frac{2}{s^2}-\frac{1}{4As^2},$$

$$\int_{t_0}^{\infty}R(s)ds=\int_{t_0}^{\infty}s^2+\sin s-\frac{2}{s^2}-\frac{1}{4As^2}ds=\infty.$$

Let

$\rho(t)=t\Rightarrow\rho'(t)=1,$

$$\lim_{t\to\infty}\int_{T_0}^{t}\frac{ds}{r(s)\rho(s)}=\lim_{t\to\infty}\int_{T_0}^{t}ds=\infty,$$

$$R(s)=\rho(s)[q(s)-p(s)]-\frac{1}{4A}\frac{(\rho'(s))^2}{\rho(s)}r(s)$$
$$=s[s+\frac{\sin s}{s}-\frac{2}{s^3}]-\frac{1}{4A}\frac{1}{s^2}.$$

For every $t\geq T_0=\frac{\pi}{2}$, we obtain

$$\int_{T_0}^{\infty}R(s)ds=\int_{T_0}^{\infty}s^2+\sin s-\frac{2}{s^2}-\frac{1}{4As^2}ds=\infty.$$

Thus, Theorem 1ensures that, every solution in this example is oscillatory.

Example 2. Let us consider the following equation

$$\left[\frac{1}{t}\left(10x'(t)+\frac{(x'(t))^{3}}{(x'(t))^{2}+1}\right)\right]'+\left(t+\frac{\cos t}{t}\right)x^3(t)=\frac{x^{5}\cos t\sin(x'(t)+2)}{(x^2+1)t^2},\ t\geq\frac{\pi}{2}$$

We note that,

$$\frac{H(t,x'(t),x(t))}{g(x(t))}=\frac{x^5\cos t\sin(x'(t)+2)}{(x^2(t)+1)t^2}\times\frac{1}{x^3(t)}\le\frac{1}{t^2}=p(t),\quad \forall x'\in\mathbb{R},x\in\mathbb{R}\ \text{ and } t\ge\frac{\pi}{2}.$$

hence

$$10<\frac{\mathrm{f(y)}}{y}=10+\frac{(x'(t))^2}{(x'(t))^2+1}<11,\ \forall y\neq 0,$$

Let

$$\rho(t)=1\Rightarrow\rho'(t)=0,$$

$$\lim_{t\to\infty}\int_{T_0}^{t}\frac{ds}{r(s)\rho(s)}=\lim_{t\to\infty}\int_{T_0}^{t}ds=\infty,$$

$$R(s)=\rho(s)[q(s)-p(s)]-\frac{1}{4A}\frac{(\rho'(s))^2}{\rho(s)}r(s),$$

$$R(s)=s+\frac{\cos s}{s}-\frac{1}{s^2},$$

$$\int_{T_0}^{\infty}R(s)ds=\infty.$$

Allof the conditions are satisfied. Hence, the differential equation in Example 2 isoscillatory.

Theorem 2. If conditionsO_1,O_2 and O_3 hold, and

O_4 $\int_{T_0}^{\infty}\rho(s)[q(s)-p(s)]ds<\infty$,

O_5 $\liminf_{t\to\infty}\left[\int_{T}^{t}R(s)ds\right]\ge 0$ for all large T,

O_6 $\lim_{t\to\infty}\int_{T_0}^{t}\frac{1}{\rho(s)r(s)}\int_{s}^{\infty}R(u)\,du\,ds=\infty$, and

O_7 $\int_{\varepsilon}^{\infty}\frac{dy}{g(y)}<\infty$ and $\int_{-\varepsilon}^{-\infty}\frac{dy}{g(y)}<\infty$ for every $\varepsilon>0$.

Then all solutions of Eq. (E_1) are oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution on $[T,\infty), T\ge T_0$ of Eq. (E_1). Without loss of generality, it is assumed that $x(t)\neq 0$.Let usassume that $x(t)$ ispositive on $[T,\infty)$ andconsider the following three cases for the behavior of $x'(t)$.

Case 1: $x'(t)>0$ for $T_1\ge T$ for some $t\ge T_1$; then from (3), we have

$$\int_{T_1}^{t}R(s)ds\ \le\frac{r(T_1)\rho(T_1)f(x'(T_1))}{g(x(T_1))}-\frac{\rho(t)r(t)f(x'(t))}{g(x(t))}$$

$$k_1\frac{r(t)\rho(t)x'(t)}{g(x(t))}\le\frac{r(T_1)\rho(T_1)f(x'(t))}{g(x(T_1))}-\int_{T_1}^{t}R(s)ds.$$

From O_1, we obtain

$$\int_{T_1}^{t}R(s)ds\le\frac{r(T_1)\rho(T_1)f(x'(T_1))}{g(x(T_1))}-k_1\frac{r(t)\rho(t)x'(t)}{g(x(t))}.$$

Hence, for all $t\ge T_1$

$$\int_{t}^{\infty}R(s)ds\le k_1\frac{r(t)\rho(t)x'(t)}{g(x(t))},$$

$$\frac{1}{r(t)\rho(t)}\int_{t}^{\infty}R(s)ds\le k_1\frac{x'(t)}{g(x(t))},$$

$$\int_{T_1}^{t}\frac{1}{r(s)\rho(s)}\int_{s}^{\infty}R(u)duds\le k_1\int_{T_1}^{\infty}\frac{x'(s)}{g(x(s))}ds,$$

$$\int_{T_1}^{t}\frac{1}{r(s)\rho(s)}\int_{s}^{\infty}R(u)duds\le k_1\int_{T_1(s)}^{\infty}\frac{dy}{g(y)}.$$

Using O_7, we obtain

$$\int_{T_1}^{t}\frac{1}{r(s)\rho(s)}\int_{s}^{\infty}R(u)duds<\infty.$$

This contradicts condition O_6.

Case 2: If $x'(t)$ is oscillatory, then there exists a sequence $\{\alpha_n\}\to\infty$ on $[T,\infty)$ such that $x'(\alpha_n)<0$. Let us assume that N is sufficiently large so that

$$\int_{\alpha_N}^{\infty}R(s)ds\ge 0.$$

Then, from O_1 and (3), we have

$$k_2\frac{\rho(t)r(t)x'(t)}{g(x(t))}\le C_{\alpha_N}-\int_{\alpha_N}^{t}R(s)ds.$$

Thus,

$$k_2\limsup_{t\to\infty}\frac{\rho(t)r(t)x'(t)}{g(x(t))}\le C_{\alpha_N}+\limsup_{t\to\infty}\left[-\int_{\alpha_N}^{t}R(s)ds\right]$$

$$= C_{\alpha_N} - \liminf_{t\to\infty}\left[\int_{\alpha_N}^{t} R(s)ds\right].$$

By O_5,we obtain

$$k_2 \limsup_{t\to\infty} \frac{\rho(t)r(t)x'(t)}{g(x(t))} < 0,$$

which contradicts the fact that $x'(t)$ oscillates.

Case 3: Let $x'(t) < 0$ for $t \ge T$ for some $T_1 \ge T$; then for any $t_0 \ge T_0$ there exists $t_1 \ge t_0$ such that $\int_{t_1}^{\infty} \rho(s)[q(s)-p(s)]ds \ge 0$ for all $t \ge t_1$. Choosing $t_1 \ge T_1$, and multiplying Eq. (E_1) by $\rho(t)$ and integrating by parts, we obtain

$$\rho(t)[r(t)f(x'(t))] - \int_{t_1}^{t} \rho'(s)r(s)f(x'(s))\,ds \le C_{t_1} - \int_{t_1}^{t} \rho(s)g(x(s))[q(s)-p(s)]ds$$

$$= C_{t_1} - g(x(t))\int_{t_1}^{t} \rho(s)[q(s)-p(s)]ds$$

$$+\int_{t_1}^{t} x'(s)g'(x(s))\int_{t_1}^{s} \rho(u)[q(u)-p(u)]duds$$

$$+\int_{t_1}^{t} \rho'(s)r(s)f(x'(s))ds\ ,$$

where $C_{t_1} = \rho(t_1)r(t_1)f(x'(t_1)) < 0.$

Thus,

$$\rho(t)r(t)f(x'(t)) \le C_{t_1}.$$

From O_1 weobtain

$$x'(t) \le \frac{C_{t_1}}{k_1}\frac{1}{r(t)\rho(t)},$$

$$x(t) \le \frac{C_{t_1}}{k_1}\int_{T_2}^{t} \frac{1}{r(s)\rho(s)} ds\ .$$

FromO_2it follows that $x(t) \to -\infty,\ as\ t \to \infty$, which is a contradiction.

Remark 1.

Condition O_5 implies that $\int_{T}^{\infty} R(s) \ge 0$ and $\liminf_{t\to\infty} \int_{T}^{\infty} R(s)ds = \int_{T}^{\infty} R(s)ds;$ hence O_6 takes the form of $\int_{T}^{\infty} R(s) \ge 0$, for all large T.

Example 3. Let us consider the following equation

$$\left[t\left(7x'(t)+\frac{(x'(t))^5}{(x'(t))^4+1}\right)\right]' + \frac{1}{t^3}x^3(t) = \frac{x^3\cos x\sin 2x'(t)}{t^4},\ t>1,$$

we note that

$$\frac{H(t,x'(t),x(t))}{g(x(t))} = \frac{x^3\cos x\sin 2x'(t)}{t^4}\times\frac{1}{x^3(t)} \le \frac{1}{t^4} = p(t),\quad \forall x'\in\mathbb{R},\ x\in\mathbb{R} \text{ and } t\ge t_0,$$

hence,

$$7<\frac{f(y)}{y} = 7+\frac{(x'(t))^4}{(x'(t))^4+1} < 8,\ \forall y\ne 0,$$

Let $\rho(t)=t \Rightarrow \rho'(t)=1$, then

$$\lim_{t\to\infty}\int_{T_0}^{t}\frac{1}{r(s)\rho(s)}ds = \lim_{t\to\infty}\int_{T_0}^{t}ds = \infty,$$

$$\int_{T_0}^{t}\rho(s)[q(s)-p(s)]ds = \int_{T_0}^{t}\left(\frac{1}{s^2}-\frac{1}{s^3}\right)ds < \infty.$$

Because,

$$R(s) = \rho(s)[q(s)-p(s)] - \frac{1}{4A}\frac{(\rho'(s))^2}{\rho(s)}r(s) = \frac{1}{s^2}-\frac{1}{s^3}-\frac{1}{4A},$$

then

$$\liminf_{t\to\infty}\int_{T}^{t}R(s)ds = \liminf_{t\to\infty}\int_{T}^{t}\left(\frac{1}{s^2}-\frac{1}{s^3}-\frac{1}{4A}\right)ds = 0,$$

$$\lim_{t\to\infty}\int_{T_0}^{t}\frac{1}{r(s)\rho(s)}\int_{s}^{\infty}R(u)du\,ds = \lim_{t\to\infty}\int_{T_0}^{t}\frac{1}{s^2}\int_{s}^{\infty}\left(\frac{1}{u^2}-\frac{1}{u^3}-\frac{1}{4A}\right)du\,ds = \infty.$$

$$\int_{\varepsilon}^{\infty}\frac{dy}{g(y)} = \int_{\varepsilon}^{\infty}\frac{dy}{y^3} = \left.\frac{-2}{y^2}\right|_{\varepsilon}^{\infty} = \frac{2}{\varepsilon^2} < \infty, \text{ and } \int_{-\varepsilon}^{-\infty}\frac{dy}{g(y)} = \frac{2}{\varepsilon^2} < \infty$$

Thus, from Theorem 2 it follows that the equation is oscillatory.

Remark 2. If we let $R(t)=1$ and $f(x'(t))=x'(t)$ in our theorems we get Theorems 1 and 2 of Remili[26] and Greaf*et al.* [10].

4.References

[1] Ayanlar, B &Tiryaki, A. 2000. Oscillation theorems for nonlinear second order differential equations with damping. *Acta Math. Hungar* 89(1-2): 1-13.

[2] Butler, G. J. 1977. The oscillatory behaviour of order nonlinear differential equations with damping. *J. Math. Anal. Appl.* 194: 1-14.

[3] Elabbasy, E. M. &Elhaddad, W. W. 2007. Oscillation of second order nonlinear differential equations with damping term. *Electronic Journal of Qualitative Theory of Differential Equations* 25: 1-19.

[4] Elabbasy, E.M. &Elzeine, Sh. R. 2011. Oscillation theorems concerning nonlinear differential equations of the second order. *Opuscula Math* 31: 373-391.

[5] Elabbasy, E.M., Hassan, S.T. &Saker, H.S. 2005. Oscillation of second-order nonlinear differential equations with damping term. *Electronic Journal of differential equations* 76: 1-13.

[6] Fang Lu & Fan Wei Mang. 2007. Oscillation theorems for superlinear second order nonlinear damped differential equations. *Applied Mathematics and Computation* 189: 796-804.

[7] Grace, S.R. &Lalli, B.S. 1990. Integral averaging technique for the oscillation of second order nonlinear differential equations. *J. Math. Anal. Appl.* 149: 277-311.

[8] Grace, S.R. 1992. Oscillation theorems for nonlinear differential equations of second order. *J. Math. Anal. Appl.* 171: 220-241.

[9] Grace, S.R. 1992. Oscillation theorems for second order nonlinear differential equations with damping. *Math. Nachr.* 141: 117-127.

[10] Greaf, J.R., Rankin, S.M. & Spikes, P.W. 1978. Oscillation theorems for perturbed nonlinear differential equations. *J. Math. Anal.Appl.* 65: 375-390.

[11] Hartman, P. 1952. Non-oscillatory linear differential equations of second order. *Amer. J.Math* 74: 389-400.

[12] Kiran, M. &Rogovchenko, V. Y. 2001. On oscillation of nonlinear second order differential equation with damping term. *Applied Mathematics and Computation* 117: 177-192.

[13] Lee, C. F.&Yeh, C. C. 2007. An Oscillation theorems. *Applied Mathematics Letters* 20: 238-240.

[14] Li, W.T. & Agarwal, R.P. 2000. Interval oscillation criteria for second order nonlinear equations with damping. *Computers Math. Applic.* 40: 217-230.

[15] Li, W.T. & Cheng, S.S. 2002. An oscillation criterion for non homogeneous half-linear differential equations. *Appl. Math. Lett.* 15: 256-263.

[16] Li, W.T. 1998. Oscillation of certain second order nonlinear differential equations, *J. Math. Anal.* 194: 1-14.

[17] Li, W.T. 2003. Interval oscillation criteria for second order nonlinear differential equations with damping. *Taiwanese Journal of Mathematics* 7: 461-475.

[18] Manojlovic, J.V. 1999. Oscillation theorems for nonlinear second order differential Equations. *Acta Sci. Math* 65: 515-527.

[19] Saad,M.J. Kumaresan, N.andKuruRatnavelu, Oscillation of Second Order Nonlinear Ordinary Differential Equations with Alternating Coefficients, Commu. in Comp. and Info.Sci. 283(2012), 367-373.

[20] Saad,M.J. Kumaresan, N.andKuruRatnavelu, Oscillation Criteria for Second Order Nonlinear Equations with Alternating Coefficients, Amer. Published in Inst.of Phys. (2013).

[21] Salhin, A. A.2014. Oscillation criteria of second order nonlinear differential equations with variable Coefficients, Discrete Dynamics in Nature and Society. 2014 (1-9).

[22]Salhin, A. A., Din, U. K. S., Ahmad, R. R. &Noorani, M. S. 2014. Some Oscillation Criteria for a Class of Second Order Nonlinear Damped Differential Equations. Appl. Math. andComput., doi: 10.1016/j.amc.2014.09.060

[23] Saad,M.J. Kumaresan, N.and KuruRatnavelu, 2019. On Oscillation of Nonlinear Differential Equations of Second Order, Sirt University Scientific Journal (Applied Sciences)Vol. 9(1). June (2019), 109-122.

[24] Salhin, A. A.2019. Oscillation Theorems for Nonlinear Second Order Forced Differential

Equations, Sirt University Scientific Journal (Applied Sciences) Vol.9 (2), December, 126-138.

[25] Saad, M. J, Salhin, A. A,and Ahmed, F. N. 2021. Oscillatory Behavior of Second Order Nonlinear Differential Equations. The 4the International Conference for Basic Sciences and Their Applications (ICVBSTA20).

[26] Remili, M. 2008. Oscillation theorem for perturbed nonlinear differential equations. *International Mathematical Forum*. 11: 513-524.

[27] Rogovchenko, V.Y &Tuncay, F. 2008. Oscillation criteria for second order nonlinear equations with damping. *Nonlinear Analysis* 69: 208-221.

[28] Rogovchenko, V.Y. 2000. Oscillation theorems for second order equations with damping. *Nonlinear Analysis* 41: 1005-1028.

[29] Saker, H.S, Pang, P.Y.H. & Agarwal, P.R. 2003. Oscillation theorems for second order nonlinear functional differential equations with damping. *Dynamic Sys.Appl*12: 307-322.

[30] Tiryaki, A. &Ayanlar, B. 2004. Oscillation theorems for certain nonlinear differential equations of second order. *Computers and Mathematics with Applications* 47: 149-159.

[31] Tiryaki, A. &Zafer, A. 2000. Oscillation criteria for second order nonlinear differential equations with damping. *Turk. J. Math.* 24:185-196.

[32] Wintner, A. 1949. A criterion of oscillatory stability. *Quart. Appl. Math.* 7: 115-117.

[33] Yeh, C.C. 1982. Oscillation theorems for nonlinear second order differential equations with damped term. *Proc. Amer. Math. Soc.* 84: 397-402.

[34] Yan, J. 1986. Oscillation theorems for second order linear differential equations with damping. *Proc. Amer. Math. Soc.* 98: 276-282.