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On a functional differential equation of first order with Riemann–Stieltjes integral and infinite point nonlocal conditions

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Abstract

In this paper, by using the Schauder fixed point theorem we study the existence of solution of the nonlocal boundary value problem of the first order functional differential equation with nonlocal conditions. The continuous dependence of the unique solution will be proved. As applications we discuss the solution of the functional differential equation with Riemann-Stieltjes integral and infinite point nonlocal conditions. Moreover, some examples illustrate importance of the results.

الملخص

الهدف من هذا البحث دراسة إثبات وجود علي الأقل حل واحد لمسالة النقطة الحدية غير المحلية للمعادلة التفاضلية الدالية من الدرجة الأولى ذات الشروط غير المحلية باستخدام نظرية القيمة الثابتة لشودر. أيضا سندرس الحل الوحيد لهذه المسألة والارتباط المستقل له.

كتطبيقات نناقش حل المعادلة التفاضلية الدالية مع شرط تكامل ريمان-ستيلجس وشرط نقطة لانهائية غير محلية. علاوة على ذلك ، توضح بعض الأمثلة أهمية النتائج.

Keywords: Functional differential equation, Riemann-Stieltjes integral condition, infinite point condition, existence of solution, continuous dependence.

1.Introduction:

II'in, et al. (1987), initiated the study of nonlocal boundary value problems of the differential equations. Some investigators have interested in studying nonlocal boundary value problem of the differential equations with nonlocal condition, in order to achieve various goals, see Byszewski, L.,(1999), Bin-Taher, E.O., (2021), El-Sayed, et al. (2021).

El-Sayed, et al. (2012) studied the existence of monotonic positive solution for the nonlocal problem the functional differential equation

$$x''(t) = f(t, x(\varphi(t))), \quad t \in (0,1),$$

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0, \quad x'(0) + \sum_{j=1}^{n} b_j x(\eta_j) = x_1$$

where $\tau_k \in (a,d) \subset (0,1)$, $\eta_j \in (c,e) \subset (0,1)$, and $x_0, x_1 > 0$.

NAJAH, et al. (2021) studied the existence and uniqueness of monotonic positive solution of the nonlocal problem

$$x''(t) = f(t, x(\phi(t)), \ t \in (0,1)$$

$$\alpha x(\tau) = x_0 > 0, \tau \in (a,d) \subset (0,1), \alpha > 0$$

$$x'(0) + \beta\, x'(\eta) = x_1 > 0, \eta \in (c,e) \subset (0,1), \beta > 0$$

In this paper, we are concerned with the nonlocal problem for the functional differential equation

$$\frac{dx}{dt} = f\left(t, x(\varphi(t))\right), \ t \in (0,T] \quad (1)$$

with the nonlocal condition

$$\sum_{k=1}^{m} a_k\, x(\tau_k) = x_0, \ \ a_k > 0, \ \tau_k \in (0,T]. \quad (2)$$

The existence of the solution $x \in C[0,T]$ of the nonlocal problem (1)-(2) will be proved. The continuous dependence of the unique solution on x_0 and a_k will be proved.

As applications, the nonlocal problem of (1) with Riemann-Stieltjes integral condition

$$\int_0^T x(s)dg(s) = x_0, \ g: [0,T] \to [0,T] \text{ is an increasing function,} \quad (3)$$

will be studied.

Also, if $\sum_{k=1}^{\infty} a_k$ is a convergent series, the nonlocal problem of (1) with infinite point nonlocal condition

$$\sum_{k=1}^{\infty} a_k\, x(\tau_k) = x_0, \ a_k > 0, \ \tau_k \in (0,T] \quad (4)$$

will be studied.

2. Main Results

2.1 Existence of solution

Let $C[0,T]$ denoted the class of continuous functions defined on $[0,T]$ with the norn given by $\|x\| = \sup_{t\in[0,T]} |x(t)|$.

And let $L^1[0,T]$ denotes the class of Lebesgue functions on $[0,T]$ with the norm given by $\|a\| = \int_0^T a(t)dt$.

We discuss the existence of the solution $x \in C[0,T]$ of the functional differential equation (1) under the following assumptions:

(i) $f: [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies Carathẽodory condition, i.e., f is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in [0,T]$.

(ii) There exist integrable function $a \in L^1[0,T]$ and a positive constant $b > 0$, such that

$$|f(t,x)| \le a(t) + b|x|, \quad \sup_{t\in[0,T]} \int_0^t a(s)ds \le M.$$

(iii) $\varphi : [0,T] \to [0,T]$ is continuous.

(iv) $2bT < 1$.

Lemma 1.

The solution of the nonlocal problem (1)-(2) can be expressed by the integral equation

$$x(t) = \frac{1}{\sum_{k=1}^m a_k}\left[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f\left(s, x(\varphi(s))\right) ds\right] + \int_0^t f\left(s, x(\varphi(s))\right) ds \quad (5)$$

Proof

Integrating (1), we get.

$$x(t) = x(0) + \int_0^t f\left(s, x(\varphi(s))\right) ds \quad (6)$$

Let $t = \tau_k$ in (6) , we have

$$\sum_{k=1}^m a_k\, x(\tau_k) = x(0)\sum_{k=1}^m a_k + \sum_{k=1}^m a_k \int_0^{\tau_k} f\left(s, x(\varphi(s))\right) ds.$$

And

$$x(0) = \frac{1}{\sum_{k=1}^m a_k}\left[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f\left(s, x(\varphi(s))\right) ds\right] \quad (7)$$

Substitute from (7) into (6), we obtain (5).

Theorem1

Let the assumptions (i)-(iv) be satisfied, then the nonlocal problem (1)-(2) has a solution $x \in C[0,T]$.

Proof

Define the set Q_r by

$Q_r = \{x \in C[0,T]: \|x\| \le r,, r > 0\} \subset C[0,T]$, where $= \frac{(\sum_{k=1}^m a_k)^{-1}|x_0| + 2M}{1-2bT}$.

Clearly, Q_r is a nonempty, closed, bounded and convex subset of $C[0,T]$.

Define the operator F related to the integral equation (5) by

$$Fx(t)\frac{1}{\sum_{k=1}^{m} a_k}[x_0-\sum_{k=1}^{m} a_k\int_0^{\tau_k} f\left(s,x(\varphi(s))\right)ds)]+\int_0^t f\left(s,x(\varphi(s))\right)ds.$$

We will show that $\{Fx\}$ is uniformly bounded on Q_r.

Let $x\in Q_r$, then we have

$$|Fx(t)|=\left|\frac{1}{\sum_{k=1}^{m} a_k}[x_0-\sum_{k=1}^{m} a_k\int_0^{\tau_k} f\left(s,x(\varphi(s))\right)ds)]+\int_0^t f\left(s,x(\varphi(s))\right)ds\right|$$

$$\le \frac{1}{\sum_{k=1}^{m} a_k}|x_0|+\int_0^{\tau_k}\left|f\left(s,x(\varphi(s))\right)\right|ds+\int_0^t\left|f\left(s,x(\varphi(s))\right)\right|ds$$

$$\le \frac{1}{\sum_{k=1}^{m} a_k}|x_0|+\int_0^{\tau_k}(a(s)+b|x|)ds+\int_0^t(a(s)+b|x|)ds$$

$$\le \frac{1}{\sum_{k=1}^{m} a_k}|x_0|+\int_0^{T}(a(s)+b|x|)ds+\int_0^T(a(s)+b|x|)ds$$

$$\le \frac{1}{\sum_{k=1}^{m} a_k}|x_0|+2M+2brT$$

$$\le r.$$

Then the class of functions $\{Fx\}$ is uniformly bounded on Q_r.

Let $x\in Q_r$ and $t_1,t_2\in(0,T]$ with $t_1<t_2$ such that $|t_2-t_1|<\delta$, then

$$|Fx(t_2)-Fx(t_1)|=\left|\int_0^{t_2} f\left(s,x(\varphi(s))\right)ds-\int_0^{t_1} f\left(s,x(\varphi(s))\right)ds\right|$$

$$=\left|\int_{t_1}^{t_2} f\left(s,x(\varphi(s))\right)ds\right|$$

$$\le\int_{t_1}^{t_2}\left|f\left(s,x(\varphi(s))\right)\right|ds$$

$$\le\int_{t_1}^{t_2}(a(s)+b|x|)ds$$

$$\le\int_{t_1}^{t_2}a(s)ds+br(t_2-t_1).$$

Thus the class of functions $\{Fx\}$ is equicontinuous on Q_r, by Arzela–Ascoli Theorem(*Kolmogorov et al.,*1970) then F is compact on Q_r.

Now, we will show that F is continuous.

Let $\{x_n\}\subset Q_r$ such that $x_n(t)\to x(t)$ in Q_r (as $n\to\infty$), i.e., this implies that $x_n(\varphi(t))\to x(\varphi(t))$ in Q_r (as $n\to\infty$) and from the continuity of the function f, we obtain

$$f\left(t,x_n(\varphi(t))\right)\to f\left(t,x(\varphi(t))\right).$$

Using assumptions (ii) and Lebesgue dominated convergence theorem (*Kolmogorov et al.*,1970) , we get

$$\lim_{n\to\infty}\int_0^{\tau_k} f\left(s,x_n(\varphi(s))\right)ds=\int_0^{\tau_k} f\left(s,x(\varphi(s))\right)ds$$

Similarly,

$$\lim_{n\to\infty}\int_0^t f\left(s,x_n(\varphi(s))\right)ds=\int_0^t f\left(s,x(\varphi(s))\right)ds.$$

And

$$\lim_{n\to\infty}(Fx_n)(t)=\lim_{n\to\infty}\frac{1}{\sum_{k=1}^m a_k}(x_0-\sum_{k=1}^m a_k\int_0^{\tau_k} f\left(s,x_n(\varphi(s))\right)ds$$

$$+\int_0^t f\left(s,x_n(\varphi(s))\right)ds$$

$$=\frac{1}{\sum_{k=1}^m a_k}\left[x_0-\sum_{k=1}^m a_k\int_0^{\tau_k} f\left(s,x(\varphi(s))\right)ds\right]+\int_0^t f\left(s,x(\varphi(s))\right)ds=Fx(t)$$

This proves that F is continuous.

Therefore, according to the Schauder fixed point theorem(*Goebel et al* 1990) there exist a solution $x\in Q_r\subset C[0,T]$ of the nonlocal problem (1)-(2).

2.2 Uniqueness of the solution

Here, we study the uniqueness of the solution $x\in C[0,T]$ of the nonlocal problem (1)-(2). For this we assume:

(v) $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is measurable in t for any $x\in\mathbb{R}$ and satisfies the Lipschitz condition with positive constant b, such that

$$|f(t,x)-f(t,y)|\le b|x-y|,$$

$$\sup_{t\in[0,T]}\int_0^t |f(s,0)|ds\le M.$$

Theorem2

Let the assumptions (iii)-(v) be satisfied. Then the solution of the nonlocal problem (1)-(2) is unique.

Proof

From assumption (v) we have

$$|f(t,x)|-|f(t,0)|\le|f(t,x)-f(t,0)|$$
$$\le b|x|,$$

Hence $|f(t,x)|\le|f(t,0)|+b|x|$.

Then all assumptions of Theorem1 are satisfied. Then there exist a solution $x\in C[0,T]$ of the nonlocal problem (1)-(2).

Let x and y be two solutions of the nonlocal problem (1)-(2). Then we have

$$|x(t)-y(t)| = \left|\frac{1}{\sum_{k=1}^{m} a_k}[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(\varphi(s))\right) ds)] + \int_0^t f\left(s, x(\varphi(s))\right) ds \right.$$
$$\left. - \frac{1}{\sum_{k=1}^{m} a_k}[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, y(\varphi(s))\right) ds)] - \int_0^t f\left(s, y(\varphi(s))\right) ds \right|$$
$$\leq \int_0^{\tau_k} \left|f\left(s, x(\varphi(s))\right) - f\left(s, y(\varphi(s))\right)\right| ds + \int_0^t \left|f\left(s, x(\varphi(s))\right) - f\left(s, y(\varphi(s))\right)\right| ds$$
$$\leq b \int_0^{\tau_k} |x(\varphi(s)) - y(\varphi(s))| ds + b \int_0^t |x(\varphi(s)) - y(\varphi(s))| ds$$
$$\leq b \int_0^{T} |x(\varphi(s)) - y(\varphi(s))| ds + b \int_0^T |x(\varphi(s)) - y(\varphi(s))| ds$$
$$\leq 2bT\|x-y\|.$$

Thus we have $(1-2bT)\|x-y\| \leq 0$.

Since $2bT < 1$, then $x(t) = y(t)$ and the solution of the nonlocal problem (1)-(2) is unique.

2.3 Continuous Dependence

Here we prove that the solution of the nonlocal problem (1)-(2) depends continuously on x_0 and on a_k.

Definition 1

The solution $x \in C[0,T]$ of the nonlocal problem (1)-(2) depends continuously on x_0, if $\forall\, \epsilon > 0$, $\exists\, \delta(\epsilon) > 0$ such that

$$|x_0 - x_0^*| \leq \delta \Longrightarrow \|x - x^*\| \leq \epsilon,$$

where x^* is the unique solution of the nonlocal problem

$$\frac{dx^*}{dt} = f\left(t, x^*(\varphi(t))\right), \quad t \in (0,T] \tag{8}$$

and $\quad \sum_{k=1}^{m} a_k\, x^*(\tau_k) = x_0^*,\ a_k > 0,\ \tau_k \in (0,T]$ (9)

Theorem3

Let the assumptions of Theorem2 be satisfied, then the solution of the nonlocal problem (1)-(2) depends continuously on x_0.

Proof

Let x and x^* be two solutions of the nonlocal problems (1)-(2) and (8) - (9), respectively, then we have

$$|x(t)-x^*(t)| = \left|\frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f\left(s, x(\varphi(s))\right) ds)] + \int_0^t f\left(s, x(\varphi(s))\right) ds \right.$$
$$\left. - \frac{1}{\sum_{k=1}^m a_k}[x_0^* - \sum_{k=1}^m a_k \int_0^{\tau_k} f\left(s, x^*(\varphi(s))\right) ds)] - \int_0^t f\left(s, x^*(\varphi(s))\right) ds \right|$$
$$\leq \frac{1}{\sum_{k=1}^m a_k}|x_0 - x_0^*| + \int_0^{\tau_k} \left|f\left(s, x(\varphi(s))\right) - f\left(s, x^*(\varphi(s))\right)\right| ds$$
$$+ \int_0^t \left|f\left(s, x(\varphi(s))\right) - f\left(s, x^*(\varphi(s))\right)\right| ds$$
$$\leq \frac{\delta}{\sum_{k=1}^m a_k} + b\int_0^{\tau_k} |x(\varphi(s)) - x^*(\varphi(s))| ds + b\int_0^t |x(\varphi(s)) - x^*(\varphi(s))| ds$$
$$\leq \frac{\delta}{\sum_{k=1}^m a_k} + 2bT\|x - x^*\|.$$

Hence $\|x - x^*\| \leq \frac{\delta}{(1-2bT)\sum_{k=1}^m a_k} = \epsilon.$

Since $2bT < 1$, then the solution of the nonlocal problem (1)-(2) depends continuously on x_0.

Definition2

The solution $x \in C[0,T]$ of the nonlocal problem (1)-(2) depends continuously on a_k, if $\forall\, \epsilon > 0, \exists\, \delta(\epsilon) > 0$ such that

$$|a_k - a_k^*| \leq \delta \implies \|x - x^*\| \leq \epsilon,$$

where x^* is the unique solution of the nonlocal problem

$$\frac{dx^*}{dt} = f\left(t, x^*(\varphi(t))\right), \quad t \in (0, T] \qquad (10)$$

and $\sum_{k=1}^m a_k^* x^*(\tau_k) = x_0, \ a_k^* > 0, \ \tau_k \in (0, T].$ (11)

Theorem4

Let the assumptions of Theorem2 be satisfied, then the solution of the nonlocal value problem (1)-(2) depends continuously on a_k.

Proof

Let x and x^* be two solutions of the nonlocal problems (1)-(2) and (10)-(11), then

$$|x(t)-x^*(t)| = \left|\frac{1}{\sum_{k=1}^m a_k}[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f\left(s, x(\varphi(s))\right) ds)] + \int_0^t \left(s, x(\varphi(s))\right) ds\right.$$

$$-\frac{1}{\sum_{k=1}^{m} a_k^*}[x_0-\sum_{k=1}^{m} a_k^*\int_0^{\tau_k} f\left(s,x^*(\varphi(s))\right)ds)]$$
$$-\int_0^t f\left(s,x^*(\varphi(s))\right)ds\Bigg|$$
$$=\Bigg|x_0\left(\frac{1}{\sum_{k=1}^{m} a_k}-\frac{1}{\sum_{k=1}^{m} a_k^*}\right)-(\frac{1}{\sum_{k=1}^{m} a_k}\sum_{k=1}^{m} a_k\int_0^{\tau_k} f\left(s,x(\varphi(s))\right)ds$$
$$-\frac{1}{\sum_{k=1}^{m} a_k^*}\sum_{k=1}^{m} a_k^*\int_0^{\tau_k} f\left(s,x^*(\varphi(s))\right)ds)$$
$$+\left(\int_0^t f\left(s,x(\varphi(s))\right)ds-\int_0^t f\left(s,x^*(\varphi(s))\right)ds\right)\Bigg|$$
$$\leq\frac{|x_0|\sum_{k=1}^{m}|a_k-a_k^*|}{\sum_{k=1}^{m} a_k\sum_{k=1}^{m} a_k^*}$$
$$+\left|\frac{\sum_{k=1}^{m} a_k^*\sum_{k=1}^{m} a_k\int_0^{\tau_k} f\left(s,x(\varphi(s))\right)ds-\sum_{k=1}^{m} a_k\sum_{k=1}^{m} a_k^*\int_0^{\tau_k} f\left(s,x^*(\varphi(s))\right)ds}{\sum_{k=1}^{m} a_k\sum_{k=1}^{m} a_k^*}\right|$$
$$+\int_0^t\left|f\left(s,x(\varphi(s))\right)-f\left(s,x^*(\varphi(s))\right)\right|ds$$
$$\leq\frac{m\delta|x_0|}{\sum_{k=1}^{m} a_k\sum_{k=1}^{m} a_k^*}+\left|\frac{\sum_{k=1}^{m} a_k\sum_{k=1}^{m} a_k^*\int_0^{\tau_k}(f\left(s,x(\varphi(s))\right)-f\left(s,x^*(\varphi(s))\right))ds}{\sum_{k=1}^{m} a_k\sum_{k=1}^{m} a_k^*}\right|$$
$$+b\int_0^t|x(\varphi(s))-x^*(\varphi(s))|ds$$
$$\leq\frac{m\delta|x_0|}{\sum_{k=1}^{m} a_k\sum_{k=1}^{m} a_k^*}+\int_0^{\tau_k}\left|f\left(s,x(\varphi(s))\right)-f\left(s,x^*(\varphi(s))\right)\right|ds+bT\|x-x^*\|$$
$$\leq\frac{m\delta|x_0|}{\sum_{k=1}^{m} a_k\sum_{k=1}^{m} a_k^*}+bT\|x-x^*\|+bT\|x-x^*\|$$
$$\leq\frac{m\delta|x_0|}{\sum_{k=1}^{m} a_k\sum_{k=1}^{m} a_k^*}+2bT\|x-x^*\|$$

Hence $\|x-x^*\|\leq\frac{m\delta|x_0|}{(1-2bT)\sum_{k=1}^{m} a_k\sum_{k=1}^{m} a_k^*}=\epsilon.$

Since $2bT<1$, then the solution of the nonlocal problem (1)-(2) depends continuously on a_k.

2.4 Nonlocal Riemann–Stieltjes integral condition

Theorem5

Let the assumptions (i)-(iv) be satisfied and let $g:[0,T]\to[0,T]$ be an increasing function. Then threre is a solution $x\in C[0,T]$ of (1) with Riemann-Stieltjes integral condition (3) and this solution is given by

$$x(t)=\frac{1}{g(T)-g(0)}[x_0-\int_0^T\int_0^{\tau_k} f\left(s,x(\varphi(s))\right)dsdg(t)]+\int_0^t f\left(s,x(\varphi(s))\right)ds.$$

Proof

Let $a_k=g(t_k)-g(t_{k-1})$, where g is an increasing function, $\tau_k\in(t_{k-1},t_k)$, and $0=t_0<t_1<t_2<\cdots<t_k=T$. Then the nonlocal condition (2) will be

$$\sum_{k=1}^{m}(g(t_k)-g(t_{k-1}))x(\tau_k)=x_0\ ,\ \sum_{k=1}^{m}a_k=g(T)-g(0).$$

And $\lim_{m\to\infty}\sum_{k=1}^{m}a_k\,x(\tau_k)=\lim_{m\to\infty}\sum_{k=1}^{m}(g(t_k)-g(t_{k-1}))\,x(\tau_k)=\int_0^T x(t)dg(t)$.

Therefore, as $m\to\infty$, the solution of the nonlocal problem (1)-(3) can be expressed as follows:

$$x(t)=\lim_{m\to\infty}\frac{1}{\sum_{k=1}^{m}a_k}[x_0-\sum_{k=1}^{m}a_k\int_0^{\tau_k} f\left(s,x(\varphi(s))\right)ds)]+\int_0^t f\left(s,x(\varphi(s))\right)ds$$

$$=\frac{1}{g(T)-g(0)}[x_0-\lim_{m\to\infty}\sum_{k=1}^{m}(g(T)-g(0))\int_0^{\tau_k} f\left(s,x(\varphi(s))\right)ds)]+\int_0^t f\left(s,x(\varphi(s))\right)ds$$

$$=\frac{1}{g(T)-g(0)}[x_0-\int_0^T\int_0^{\tau_k} f\left(s,x(\varphi(s))\right)dsdg(t)]+\int_0^t f\left(s,x(\varphi(s))\right)ds.$$

2.5 Infinite point nonlocal condition

Theorem6

Let the assumptions (i)-(iv) be satisfied and let $\sum_{k=1}^{\infty}a_k$ be a convergent series. Then threre is a solution $x\in C[0,T]$of (1) with infinite point nonlocal condition (4) and this solution is given by

$$x(t)=\frac{1}{\sum_{k=1}^{\infty}a_k}[x_0-\sum_{k=1}^{\infty}a_k\int_0^{\tau_k} f\left(s,x(\varphi(s))\right)ds)]+\int_0^t f\left(s,x(\varphi(s))\right)ds \quad (12)$$

Proof

For any positive integer m, the solution of the nonlocal problem (1)-(2)

can be expressed as follows:

$$x_m(t) = \frac{1}{\sum_{k=1}^m a_k}\left[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f\left(s, x_m(\varphi(s))\right) ds\right] + \int_0^t f\left(s, x_m(\varphi(s))\right) ds \qquad (13)$$

Take the limit to (13) as $m \to \infty$, then we have

$$\lim_{m\to\infty} x_m(t) = \lim_{m\to\infty} \frac{1}{\sum_{k=1}^m a_k}\left[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f\left(s, x_m(\varphi(s))\right) ds\right] + \lim_{m\to\infty} \int_0^t f\left(s, x_m(\varphi(s))\right) ds$$

Now,

$$|a_k x(\tau_k)| \le a_k \|x\| \le a_k r.$$

And

$$\left|a_k \int_0^{\tau_k} f\left(s, x_m(\varphi(s))\right) ds\right| \le a_k \int_0^{\tau_k} \left|f\left(s, x_m(\varphi(s))\right)\right| ds \le a_k \int_0^{\tau_k} (a(s) + b\|x_m\|) ds \le a_k(M + brT).$$

Therefore, by comparison test the series $\sum_{k=1}^m a_k\, x(\tau_k)$ and $\sum_{k=1}^\infty a_k \int_0^{\tau_k} f\left(s, x_m(\varphi(s))\right) ds$ are convergent.

Using assumption (ii) and Lebesgue Dominated Theorem (*Kolmogorov et al.*,1970) , from (13) we obtain

$$x(t) = \frac{1}{\sum_{k=1}^\infty a_k}\left[x_0 - \sum_{k=1}^\infty a_k \int_0^{\tau_k} f\left(s, x(\varphi(s))\right) ds\right] + \int_0^t f\left(s, x(\varphi(s))\right) ds.$$

Also, we have

$$\sum_{k=1}^\infty a_k\, x(\tau_k) = \sum_{k=1}^\infty a_k \frac{1}{\sum_{k=1}^\infty a_k}\left[x_0 - \sum_{k=1}^\infty a_k \int_0^{\tau_k} f\left(s, x(\varphi(s))\right) ds\right] + \int_0^{\tau_k} f\left(s, x(\varphi(s))\right) ds = x_0$$

This proves that the solution of (12) satisfies (1) under infinite point condition (4).

2.6 Examples.

In this section, we provide some examples to illustrate our results.

Example1

Consider the following functional differential equation

$$\frac{dx}{dt} = t^4 e^{-t} + \frac{1}{17}\left[e^{-t^2} x(t^2) cos^2\left(x(t^2)\right) + t^4 x(t^2)\right], \quad t \in (0,1] \qquad (14)$$

with the nonlocal integral condition

$$\int_0^1 x(s)dg(s) = x_0, \qquad (15)$$

Here, $\varphi(t) = t^2$ is a continuous function

Now set

$$f\left(t, x(\varphi(t))\right) = t^4 e^{-t} + \frac{1}{17}\left[e^{-t^2} x(t^2) cos^2\left(x(t^2)\right) + t^4 x(t^2)\right].$$

Then $\left|f\left(t, x(\varphi(t))\right)\right| \le t^4 e^{-t} + \frac{2}{17}|x|.$

The assumptions (i)–(iv) of Theorem1 are satisfied ,with $b = \frac{2}{17}$,

$a(t) = t^4 e^{-t} \in L^1[0,1]$ and $2bT \simeq 0.24 < 1.$

Also the function $g: [0,1] \to [0,1]$ defined by $g(t) = t$ is increasing.

Therefore, from Theorem5, the nonlocal problem (14)–(15) has a solution $x \in C[0, 1]$.

Example2

Consider the following functional differential equation

$$\frac{dx}{dt} = t^3 + 2t + \frac{ln(1+|x(\beta t)|)}{15-t}, \quad t \in (0, 3] \qquad (16)$$

with infinite point nonlocal condition

$$\sum_{k=1}^{\infty} \frac{1}{k^2} x\left(\frac{3k-1}{k^2+1}\right) = x_0, \qquad (17)$$

where $\in (0, 1)$, here we have $\varphi(t) = \beta t$ which is acontinuous function.

If we set

$$f\left(t, x(\varphi(t))\right) = t^3 + 2t + \frac{ln(1+|x(\beta t)|)}{15-t}.$$

Then $\left|f\left(t, x(\varphi(t))\right)\right| \le t^3 + 2t + \frac{1}{12}|x|.$

The assumptions (i) – (iv) of Theorem1 are satisfied, with $b = \frac{1}{12}$,

$a(t) = t^3 + 2t \in L^1[0,1]$, and $2bT \simeq 0.5 < 1.$

Therefore, from Theorem6, the nonlocal problem(16)–(17) has a solution $x \in C[0, 3]$.

3.Conclusion

In this paper, we introduce a nonlocal boundary value problem for functional differential equations with nonlocal conditions . Here we have proved the existence of solution for the nonlocal problem (1)–(2). The sufficient conditions for the uniqueness have been given and the continuous dependence has been proved. Generalization for the nonlocal condition (2) to (3) and (4) has been proved. Some examples to illustrate the obtained results have been given.

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