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On Neighborhoods of Analytic Function Subclasses Associated with the q–Ruscheweyh Operator

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Abstract

This Abstract: In this present, we extend the notion of neighborhoods in families of analytic functions using q-derivative . A systematic study of inclusion relationships among (m, δ, q)-neighborhoods of canonically normalized analytic functions with negative coefficients is presented

Keywords
Analytic functions, neighborhoods, *q*-derivative

الجوارات لفصول جزئية للدوال التحليلية المتضمنة مؤثر q راشويه

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الملخّص

في هذا البحث، نقوم بتوسيع مفهوم الجوار للدوال التحليلية من خلال استخدام المشتقة q. كما نثبت عدة علاقات تتعلق بالاحتواء المرتبط بالجوارات لبعض الفئات الفرعية للدوال التحليلية التي تحتوي على معاملات سالبة.

الكلمات المفتاحية
دوال حليلية
جوارات–
المشتقة .

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1. Introduction and Definitions:

Denote by $A(m)$ the class of functions consisting of functions Ω of the form

$$\Omega\,(\xi) = \xi - \sum_{r=m+1}^{\infty} \sigma_r\, \xi^r \;\; (\sigma_r \geq 0; \quad m \in \mathbb{N} := \{1,2, \ldots\}), \tag{1.1}$$

which are analytic in the open unit disc U.

In recent years q-calculus has motivated the researchers a lot due to its various applications in Mathematics and Physics. The application of q-calculus was initiated by Jackson (1908;1910). He was the first to develop q-integral and q-derivative. Using convolution of normalized analytic functions, Aldweby and Darus (2014) defined q-analogue of Ruscheweyh differential operator and studied some of its properties. In recent years, numerous authors have explored novel classes of analytic functions by employing q-calculus operators. For some of the latest studies on these classes and related topics, readers may consult the following references: (Aldweby & Darus, 2013; 2014; Seoudy & Aouf, 2016).

For a function $\Omega \in A(m)$ given by (1.1), the q-derivative D_q of a function Ω is defined by (Jackson 1908;1910).

$$D_q\Omega(\xi) = \frac{\Omega(q\xi) - \Omega(\xi)}{(q-1)\xi} \qquad (\xi \neq 0, 0 < q < 1) \tag{1.2}$$

and $D_q\Omega(0) = \Omega'(0)$ and $D_q^2\Omega(\xi) = D_q\left(D_q\Omega(\xi)\right)$. From (1.2), we deduce that

$$D_q(\Omega(\xi)) = 1 - \sum_{r=m+1}^{\infty} [r]_q \sigma_r\, \xi^{r-1},$$

where

$$[r]_q = \frac{1-q^r}{1-q} = 1 + q + \cdots + q^r \longrightarrow r. \tag{1.3}$$

As $q \longrightarrow 1^-, [r]_q \to r,$ for a function $\Lambda(\xi) = \xi^r$, we get

$$D_q(\xi^r) = [r]_q \xi^{r-1}.$$

$$\lim_{q \to 1^-} (D_q(\xi^r)) = r\xi^{r-1} = \Lambda'(\xi)$$

where Λ' is the standard derivative.

Employing the q-derivative of a function $\Omega \in A(m)$, we develop a generalized framework for q-neighborhoods of analytic functions which is generalizes the neighborhood defined by Goodman (1957) and Ruscheweyh (1981).

Definition 1.1 Let $\Omega \in A(m)$ of the form (1.1) and $\delta \geq 0$, we define the (m, δ, q)-neighborhood of Ω as follows:

$$N_{m,\delta,q}(\Omega) := \left\{ \Lambda \in \mathrm{A(m)}: \Lambda(\xi) = \xi - \sum_{r=m+1}^{\infty} \tau_r \, \xi^r \; and \; \sum_{r=m+1}^{\infty} [r]_q \, |\sigma_r - \tau_r| \leq \delta \right\} \qquad (1.4)$$

For the identity function $e(\xi) = \xi$, we have

$$N_{m,\delta,q}(\mathrm{e}) := \left\{ \Lambda \in \mathrm{A(m)}: \Lambda(\xi) = \xi - \sum_{r=m+1}^{\infty} \tau_r \, \xi^r \; and \; \sum_{r=m+1}^{\infty} [r]_q \, |\tau_r| \leq \delta \right\} \qquad (1.5)$$

A function $\Omega \in A(m)$ is said to be q-starlike of order α, that is, $S_q^*(\alpha, m)$ if it fulfills the following statment

$$\mathrm{Re}\left(\frac{\xi \mathrm{D_q}(\Omega(\xi))}{\Omega(\xi)} \right) > \alpha, \qquad \xi \in \mathrm{U}, \qquad 0 < \mathrm{q} < 1$$

Beyond this , a function $\Omega \in A(m)$ is belong to the q-convex of order α, $C_q(\alpha, m)$

$$Re\left(\frac{D_q(\xi \, D_q(\Omega(\xi)))}{D_q \Omega(\xi)} \right) > \alpha, \qquad 0 \leq \alpha < 1, \qquad 0 < q < 1, \qquad \xi \in U.$$

The classes $S_q^*(\alpha, m)$ and $C_q(\alpha, m)$ established by Seoudy and Aouf (2016) and Aldweby And Darus (2016).

To proceed, we recall the q- Ruscheweyh operator established by Aldweby and Darus (2014), which plays a fundamental role in our results.

Definition1.2: If $\Omega \in A(m)$, the q- Ruscheweyh operator expressed by

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$$R_q^\lambda \Omega(\xi) = \xi + \sum_{r=m+1}^{\infty} \frac{[r+\lambda-1]_q!}{[\lambda]_q!\,[r-1]_q!}\,\sigma_r\,\xi^r \qquad (1.6)$$

where $[r]_q!$ given by

$$[r]_q! = \begin{cases} [r]_q[r-1]_q \dots [1]_q, & r = 1,2,\dots, \\ 1\,, & r = 0. \end{cases} \qquad (1.7)$$

For $q \longrightarrow 1$, we get

$$\lim_{q\to 1} R_q^\lambda \Omega(\xi) = \xi + \lim_{q\to 1}\left[\sum_{r=m+1}^{\infty} \frac{[r+\lambda-1]_q!}{[\lambda]_q!\,[r-1]_q!}\,\sigma_r\,\xi^r\right]$$

$$= \xi + \sum_{r=m+1}^{\infty} \frac{(r+\lambda-1)!}{(\lambda)!\,(r-1)!}\,\sigma_r\,\xi^r = R^\lambda \Omega(\xi),$$

Since $R^\lambda \Omega(\xi)$ established in (Ruscheweyh, 1975).

Building upon the concept of the q-derivative operator, we define the following classes of q-starlike and q-convex functions in U.

Definition 1.3: Let $\Omega \in A(m)$, , $0 \le \alpha < 1$ and $\lambda > -1$. Then $\Omega \in S_{q,\lambda}^*(\alpha, m)$ if it satisfies the following inequality:

$$Re\left(\frac{\xi D_q(R_q^\lambda \Omega(\xi))}{R_q^\lambda \Omega(\xi)}\right) > \alpha, \qquad \xi \in U, \qquad 0 < q < 1$$

where R_q^λ is defined by (1.6). Also $\Omega \in C_{q,\lambda}(\alpha, m)$ *if* it satisfies the inequality

$$Re\left(\frac{D_q\left(\xi D_q\left(R_q^\lambda \Omega(\xi)\right)\right)}{D_q(R_q^\lambda \Omega(\xi))}\right) > \alpha, \qquad \xi \in U, \qquad 0 < q < 1.$$

Noting that

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$$\lim_{q\to 1} S^*_{q,0}(\alpha,1)=\left\{\Omega\in A:\ \lim_{q\to 1} Re\left(\frac{\xi D_q(R_q^{\lambda}\Omega(\xi))}{R_q^{\lambda}\Omega(\xi)}\right)>\alpha,\xi\in U\right\}=S^*(\alpha),$$

$$\lim_{q\to 1} C_{q,0}(\alpha,1)=\left\{\Omega\in A:\ \lim_{q\to 1}\ Re\left(\frac{D_q(\xi D_q\left(R_q^{\lambda}\Omega(\xi)\right))}{D_q(R_q^{\lambda}\Omega(\xi))}\right)>\alpha,\xi\in U\right\}=C(\alpha),$$

In this context, $S*(\alpha)$ and $C(\alpha)$ refer to the standard classes of starlike functions of order α and convex functions of order α in U(see Ma & Minda (1992)).

2. Analytic Properties for $\mathbf{N_{m,\delta,q}(e)}$:

To establish inclusion relations involving the (m,δ, q) neighborhood ,we first prove the following key lemma:

Lemma 2.1 Let $\Omega\in A(m)$ given by (1.1). Then $\Omega\in S^*_{q,\lambda}(\alpha,m)$ if and only if

$$\sum_{r=m+1}^{\infty}([r]_q-\alpha)\frac{[r+\lambda-1]_q!}{[\lambda]_q!\,[r-1]_q!}|\sigma_r|\le 1-\alpha,\qquad 0\le\alpha<1 \tag{2.1}$$

Further, $\Omega\in C_{q,\lambda}(\alpha,m)$ if and only if

$$\sum_{r=m+1}^{\infty}[r]_q(q[r-1]_q+(1-\alpha))\frac{[r+\lambda-1]_q!}{[\lambda]_q!\,[r-1]_q!}|\sigma_r|\le 1-\alpha,\qquad 0\le\alpha<1 \tag{2.2}$$

We begin by characterizing the inclusion properties of $N_{m,\delta,q}(e)$:

Theorem 2.2 Let

$$\delta=\frac{(1-\alpha)(1+q[m]_q)[\lambda]_q!\,[m]_q!}{(1-\alpha+q[m]_q)[m+\lambda]_q!}$$

Where $\alpha\in[0,1), q\in(0,1), \lambda>-1, m\in\mathbb{N}$. Then

$$S^*_{q,\lambda}(\alpha,m)\subset N_{m,\delta,q}(e).$$

Proof. For $\Omega\in S^*_{q,\lambda}(\alpha,m)$ and making use of the condition (2.1), we obtain

$$([m+1]_q-\alpha)\frac{[m+\lambda]_q!}{[\lambda]_q!\,[m]_q!}\sum_{r=m+1}^{\infty}\sigma_r\le 1-\alpha. \tag{2.3}$$

On the other hand, we also find from (2.1) and (2.3) that

$$\frac{[m+\lambda]_q!}{[\lambda]_q!\,[m]_q!}\sum_{r=m+1}^{\infty}([r]_q-\alpha)\sigma_r\le 1-\alpha. \tag{2.4}$$

$$\frac{[m+\lambda]_q!}{[\lambda]_q!\,[m]_q!}\sum_{r=m+1}^{\infty}[r]_q\sigma_r\le 1-\alpha+\frac{[m+\lambda]_q!}{[\lambda]_q!\,[m]_q!}\alpha\sum_{r=m+1}^{\infty}\sigma_r.$$

Hence,

$$\sum_{r=m+1}^{\infty}[r]_q.\sigma_r\le\frac{(1-\alpha)\left(1+q[m]_q\right)[\lambda]_q!\,[m]_q!}{\left(1-\alpha+q[m]_q\right)[m+\lambda]_q!}=\delta$$

Which in view of (1.5), proves Theorem 2.2.

Remark 2.1 If $\lambda=1$, we obtain that

$$\sum_{r=m+1}^{\infty}[r]_q.\sigma_r\le\frac{(1-\alpha)\left(1+q[m]_q\right)}{\left(1-\alpha+q[m]_q\right)[m+1]_q}=\delta$$

for $R_q^1\Omega=\xi D_q\Omega,\ \ \Omega\ \in S_q^*(\alpha,m)$.

By employing the same method with (2.2) substituted for (2.1), the proof technique yields the following immediate consequence.

Corollary 2.3 Let

$$\delta=\frac{(1-\alpha)[\lambda]_q!\,[m]_q!}{\left(1-\alpha+q[m]_q\right)[m+\lambda]_q!}$$

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where $\alpha \in [0,1), q \in (0,1), \lambda > -1, m \in \mathbb{N}$. Then

$$C_{q,\lambda}(\alpha, m) \subset N_{m,\delta,q}(e).$$

Remark 2.2 If $\lambda = 1$, we obtain that

$$\sum_{r=m+1}^{\infty} [r]_q . \sigma_r \leq \frac{(1-\alpha)}{(1-\alpha+q[m]_q)[m+1]_q} = \delta,$$

for $R_q^1 \Omega = \xi D_q \Omega, \Omega \,\epsilon C_q(\alpha, m)$.

3. Neighborhoods for the Classes $S_{q,\lambda}^{*(\beta)}(\alpha, m)$ and $C_{q,\lambda}^{(\beta)}(\alpha, m)$:

Our investigation focuses on the neighborhood properties of the following q-calculus function classes

A function $\Omega \in A(m)$ is belong to the class $S_{q,\lambda}^{*(\beta)}(\alpha, m)$ if for some function $\Lambda \in S_{q,\lambda}^{*}(\alpha, m)$ such that

$$\left|\frac{\Omega(\xi)}{\Lambda(\xi)} - 1\right| < 1 - \beta \quad (\xi \epsilon\, U; 0 \leq \beta < 1) \tag{3.1}$$

Analogously, a function $\Omega \in A(m)$ belongs to the class $C_{q,\lambda}^{(\beta)}(\alpha, m)$ if for some function $\Lambda \in C_{q,\lambda}(\alpha, m)$ where the inequality (3.1) holds true.

Theorem 3.1 If $\Lambda \in S_{q,\lambda}^{*}(\alpha, m)$ and

$$\beta = 1 - \frac{\delta . \left[\left(q^m + [m]_q - \alpha\right) \frac{[m+\lambda]_q!}{[\lambda]_q! \, [m]_q!}\right]}{\left(q^m + [m]_q\right) . \left[\left(q^m + [m]_q - \alpha\right) \frac{[m+\lambda]_q!}{[\lambda]_q! \, [m]_q!} - (1-\alpha)\right]} \tag{3.2}$$

then

$$N_{m,\delta,q}(\Lambda) \subset S_{q,\lambda}^{*(\beta)}(\alpha, m)$$

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Proof. Let $\Omega \in N_{m,\delta,q}(\Lambda)$, we then find from the Definition 1.1 that

$$\sum_{r=m+1}^{\infty} [r]_q |\sigma_r - \tau_r| \leq \delta,$$

which readily implies the coefficients inequality:

$$\sum_{r=m+1}^{\infty} |\sigma_r - \tau_r| \leq \frac{\delta}{q^m + [m]_q}.$$

Since, $\Lambda \in S^*_{q,\lambda}(\alpha, m)$, we have

$$\sum_{r=m+1}^{\infty} \tau_r \leq \frac{1-\alpha}{(q^m + [m]_q - \alpha)\frac{[m+\lambda]_q!}{[\lambda]_q!\,[m]_q!}}.$$

So that,

$$\left|\frac{\Omega(\xi)}{\Lambda(\xi)} - 1\right| < \frac{\sum_{r=m+1}^{\infty}|\sigma_r - \tau_r|}{1 - \sum_{r=m+1}^{\infty}\tau_r}$$

$$\leq \frac{\delta.\left[(q^m + [m]_q - \alpha)\frac{[m+\lambda]_q!}{[\lambda]_q!\,[m]_q!}\right]}{(q^m + [m]_q).\left[(q^m + [m]_q - \alpha)\frac{[m+\lambda]_q!}{[\lambda]_q!\,[m]_q!} - (1-\alpha)\right]} = 1 - \beta$$

provided that β is given by (3.2). Thus, from (3.1), we have , $\Omega \in S^{*(\beta)}_{q,\lambda}(\alpha, m)$ for β specified by (3.2). Hence, the argument is complete.

Theorem 3.2 If $\Lambda \in C_{q,\lambda}(\alpha, m)$ and

$$\beta = 1 - \frac{\delta.\left[[m+1]_q(q^m + [m]_q - \alpha)\frac{[m+\lambda]_q!}{[\lambda]_q!\,[m]_q!}\right]}{([m+1]_q).\left[[m+1]_q(q^m + [m]_q - \alpha)\frac{[m+\lambda]_q!}{[\lambda]_q!\,[m]_q!} - (1-\alpha)\right]} \qquad (3.3)$$

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then

$$N_{m,\delta,q}(\Lambda) \subset C_{q,\lambda}^{(\beta)}(\alpha, m)$$

Proof. Let $\Omega \in N_{m,\delta,q}(\Lambda)$, we then find from the Definition 1.1 that

$$\sum_{r=m+1}^{\infty} [r]_q |\sigma_r - \tau_r| \leq \delta,$$

Consequently, we derive the following coefficient estimate

$$\sum_{r=m+1}^{\infty} |\sigma_r - \tau_r| \leq \frac{\delta}{q^m + [m]_q}.$$

Since, $\Lambda \in C_{q,\lambda}(\alpha, m)$, we have

$$\sum_{r=m+1}^{\infty} \tau_r \leq \frac{1-\alpha}{[m+1]_q \left(q^m + [m]_q - \alpha\right) \frac{[m+\lambda]_q!}{[\lambda]_q! \, [m]_q!}}.$$

So that,

$$\left|\frac{\Omega(\xi)}{\Lambda(\xi)} - 1\right| < \frac{\sum_{r=m+1}^{\infty} |\sigma_r - \tau_r|}{1 - \sum_{r=m+1}^{\infty} \tau_r}$$

$$\leq \frac{\delta.\left[[m+1]_q \left(q[m]_q + 1 - \alpha\right) \frac{[m+\lambda]_q!}{[\lambda]_q! \, [m]_q!}\right]}{\left([m+1]_q\right).\left[[m+1]_q \left(q[m]_q + 1 - \alpha\right) \frac{[m+\lambda]_q!}{[\lambda]_q! \, [m]_q!} - (1-\alpha)\right]} = 1 - \beta$$

provided that β is given by (3.3). Thus, from (3.1), we have , $\Omega \in C_{q,\lambda}^{(\beta)}(\alpha, m)$ for β specified by (3.3). Hence, the argument is complete.

Remark 3.1 Taking $\lambda = 1$ and $q \to 1$ in the above results, we can derive the corresponding inclusion relations obtained by Altinta and Owa (1996).

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