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The product and free product of fuzzy subgroups

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Abstract: The free product of fuzzy sets in a group theory is such attractive topic for many researchers. This product, which lies between the sup-min and sup-max products, has number of important properties and also some potential applications. It is also demonstrated that, the standard results for the product of two subgroups in a group are simulated by the free product. The employing of previous concepts and results within the fuzzy set framework is made possible by its relationship to group theory. Here, in this paper, we shall study the free product of two fuzzy sets in a groupoid in order to obtain some results regarding to the subject of the free product.

Key words: Free Product, Groupoid, Fuzzy Sets, Subgroups.

الملخــص.يعتبر الضرب الحر للمجموعات الضبابية في نظرية الزمر موضوعا جذابا للكثير من الباحثين. هذا النوع من عمليات الضرب والتي تحدث بين القيم العليا والدنيا له عدد من الخصائص الهامة وبعض التطبيقات المحتملة. وقد ثبت أيضا أنه بالإمكان محاكاة النتائج القياسية لضرب زمرتين جزئيتين في زمرة جزئية بواسطة الضرب الحر. إن توظيف النتائج والمفاهيم السابقة ضمن إطار المجموعة الضبابية أصبح ممكنا من خلال علاقتها بنظرية المجموعات. هنا، في هذه البحث، سندرس الضرب الحر لمجموعتين ضبابيتين في زمرة شكلية (groupoid) من أجل الحصول على بعض النتائج المتعلقة بموضوع الضرب الحر.

1.Introduction:

The concept of fuzzy sets was introduced by Zadeh (1965). Since then, the theory of fuzzy sets has developed in many different applications in a wide variety of fields. The idea behind fuzzy set theory was that classical sets were not appropriate concept in expressing the difficulties and challenges experienced in everyday life. The reason of that, everything encountered in this real physical world has some degree of fuzziness.

The ideas of fuzzy subgroupoid and fuzzy subgroup were introduced by Rosenfeld (1971), which opened the beginning of the study of fuzzy group theory. Thus, the foundation for further study of various kinds of fuzzy algebraic substructures was established. The distinguishing between fuzzy subgroups and ordinary subgroups is that fuzzy subgroups are characterized by the inability to identify with certainty which group elements belong and which do not.

However, Liu (1982) first proposed the idea of the product of fuzzy sets in groups. Since then, it has been extensively studied and applied across various branches of fuzzy algebraic structures. This work established the foundation for the use of fuzzy algebraic structures by improving on previous developments in fuzzy set theory. The notion of the product of fuzzy sets achieved widespread acceptance in the field of fuzzy mathematics following Liu's key work in this area. Fuzzy set products have been found applications in many branches of fuzzy algebraic structures since Liu's work such as fuzzy groups, fuzzy semigroup, fuzzy rings, fuzzy modules, fuzzy vector space and other fuzzy algebraic systems. The product of fuzzy sets in groups allows for the study of the algebraic properties of fuzzy sets under different binary operations. This has enabled researchers to develop a rich theory of fuzzy algebraic structures, with numerous results and applications in areas like coding theory, decision-making, image processing and pattern recognition. See Ajmal & Thomas (1993) and Palaniappan (2005).

The concept of a free product is presented and covered in details by Ray (1992). In the theory of fuzzy groups, the free product of fuzzy subgroups is a construction that permits the combining of several fuzzy subgroups into a larger fuzzy group. In the study of the algebraic structure of fuzzy sets, it is a crucial idea. The idea of appointing the membership degree to be the maximum of the minimum membership degrees of the factors that is, taking all possible products of elements from the component fuzzy subgroups is captured in this formulation.

In the following, let we review several fundamental definitions and findings, which will be used later on. For more information regarding to the development in this field of study, one can see papers of [Ajmal, 2000; Ajmal & Jahan, 2012; Atif & Mishref, 1996; Bayrak & Yamak, 2016; Dixit, 1981]. Also, one can see the resent papers of [Awolola & Ibrahim, 2021; Bejines et. al., 2021; Gayen et. al., 2019] and references therein.

Preliminaries

In here, we recall some important definitions:

Definition 1.1 [Dummit & Foote, 2004]: A group $(G,*)$ is called a permutation group on a nonempty set X if the elements of G are permutations of X and the operation $*$ is the composition of two functions.

Definition 1.2 [Dummit & Foote, 2004]: Let $I_n = \{1, 2, \dots., n\}, n \geq 1$. Let π be a permutation on I_n. Then

$$\pi = \{(1, \pi(1)), (2, \pi(2)), \dots \dots\ (n, \pi(n))\}.$$

Definition 1.3 [Zadeh, 1965]: Let X be non-empty set. A fuzzy subset μ of the set X is a function $\mu: X \to [0,1]$.

Definition 1.4 [Zadeh, 1965]: A fuzzy subgroup μ of a group G is called improper if μ is constant on the group G and is called proper if μ have at least two values in the interval $[0,1]$.

Definition 1.5 [Das, 1981]: Let G be a group and μ be a fuzzy subgroup of G. The subgroups $\mu_t, t \in [0, \mu(e)]$ are called level subgroups of μ.

Definition 1.6 [Ray, 1992]: A fuzzy subgroup μ of G is called fuzzy normal (invariant) if $\mu(xy) = \mu(yx)$ for $x, y\ in\ G$.

Definition 1.7 [Liu, 1982]: let η and μ be fuzzy sets in a groupoid G. Then, the set product $\eta \circ \mu$ is the fuzzy set in G, defined by:

$$\eta \circ \mu = \begin{cases} \sup\{\min(\eta(x(,\mu(y)\} & if\ g = xy\ for\ some\ x,y \epsilon G \\ \quad g = xy & \\ \quad o & oterwise. \end{cases}$$

Liu (1982) introduced and analyzed $\eta \circ \mu$ and recognized as standard fuzzy product.

Next, we present our result.

Theorem1.1: let η and μ be fuzzy subgroups of a group G. Then the set product $\eta_{\circ}\mu$ contains η and μ if and only if η and μ have the same tip, that is $\eta(e) = \mu(e)$.

Proof: To prove that, we have to prove the following inequalities:

$$\eta \leq \eta \circ \mu \quad \leftrightarrow \quad \eta(e) \leq \mu(e) \tag{1}$$

$$\mu \leq \eta \circ \mu \quad \leftrightarrow \quad \mu(e) \leq \eta(e) \tag{2}$$

Let start by (1) by supposing that

$$\eta \leq \eta \circ \mu,$$

then

$$\eta \circ \mu = \begin{cases} \sup\{\min(\eta(x),\mu(x^{-1}))\} \\ \quad x \in G \end{cases}$$

$$\leq \min(\eta(e),\mu(e))$$

Therefore, if $\eta(e) > \mu(e)$, then $\eta \circ \mu(e) = \mu(e) < \eta(e)$ and hence $\eta \nleq \eta \circ \mu$ but this is a contradiction.

Conversely, suppose $\eta(e) \leq \mu(e)$ and let g$\in G$. Then

$$\eta \circ \mu(g) = \sup_{g=xy}\{\min\{(\eta(x),\mu(y)\}\}$$

$$\geq \min\{\eta(g(,\mu(e)\}$$

$$\geq \min\{\eta(g), \mu(e), \ \eta(e) \leq \mu(e)$$

$$= \eta(g).$$

Now, from (2), we suppose that

$$\eta \circ \mu \geq \mu,$$

then

$$\eta \circ \mu(e) \ = \sup_{x \in G}\{\min(\eta(e), \mu(x^{-1}))\}$$

$$\leq \min(\eta(e), \mu(e))$$

So if $\mu(e) \ > \eta(e)$ then $\eta \circ \mu(e) \ = \eta(e) \ < \mu(e)$ and hence $\mu \nleq \eta \circ \mu$ but this is a contradiction.

Conversely**,** uppose $\mu(e) \leq \eta(e)$ and let $g \in G$. Then

$$\eta \circ \mu(g) \ = \ \sup_{g=xy}\{(\eta(x), \mu(y))\}$$

$$\geq \min(\eta(e), \mu(g))\}$$

$$\geq \min(\eta(e), \mu(g)\} \text{ if } \mu(e) \leq \eta(e)$$

$$= \mu(g)$$

From (1) and (2) we arrive at

$$\eta \leq \eta \circ \mu, \ \mu \leq \eta \circ \mu \leftrightarrow \eta(e) = \mu(e).$$

Definition 1.8 Let η and μ be fuzzy sets in a groupoid G with supports A and B, respectively. Then, for $g \in G$, we define subsets

$A(g, L, \mu) = \{x \in A: xy = g$ for some $y \in B\}$,

$B(g, R, \eta) = \{y \in B: xy = g$ for some $x \in A\}$,

$A(g, R, \mu) = \{x \in A: yx = g$ for some $y \in B\}$,

$B(g, L, \eta) = \{y \in B: yx = g$ for some $x \in A\}$, A fuzzy set $\eta * \mu$ is called the free product of η and μ. As a consequence of the definition of free product it is obvious that if the supports of fuzzy subgroups η and μ in a group G coincide with G then $\eta * \mu$ assumes a constant value on G.

The following example shows that the product of subgroup is subgroup.

Example 1.1: Let $G = S_4$, the symmetric group of degree 4, the fuzzy subgroups μ and η are defined as follows:

$\eta(e) = \frac{6}{7}$,

$\eta(\{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}) = \left\{\frac{4}{7}\right\}$,

$\eta(\gamma) = 0$ for any other permutation γ in G,

$\mu(\{e, (12)\}) = \left\{\frac{5}{7}\right\}, \mu(\{(12)(34), (34)\}) = \left\{\frac{3}{7}\right\}$,

$\mu(\{(13)(24), (14)(23), (1234), (1423)\}) = \left\{\frac{1}{7}\right\}$,

$\mu(\beta) = 0$ for any other permutation β in G.

Keeping all the level subgroups of intact and slightly changing the values of range sets Im μ and Im η, we have new fuzzy subgroups $\bar{\mu}$ and $\bar{\eta}$ as follows:

$$\bar{\eta}(\alpha) = \eta(\alpha) \quad for \quad \alpha \in \eta_{2/7}$$

$$\bar{\eta}(\alpha) = \frac{1}{7} \quad for \quad \alpha \in G \sim \eta_{2/7}.$$

$$\bar{\mu}(\alpha) = \mu(\alpha) \quad for \quad \alpha \in \mu_{3/7},$$

$$\bar{\mu}(\alpha) = \frac{2}{7} \quad for \quad \alpha \in \mu_{1/7} \sim \mu_{3/7},$$

$$\bar{\mu}(\alpha) = \frac{1}{7} \quad for \quad \alpha \in G \sim \mu_{1/7}.$$

Here $\bar{\mu}$ is equivalent to μ and $\bar{\eta}$ is equivalent to η, that is $\bar{\mu} \in [\mu]$ and $\bar{\eta} \in [\mu]$. Let us compare the free product of $\bar{\mu}$ and $\bar{\eta}$ with their set product.

Free product: The supports of the both $\bar{\eta}$ and $\bar{\mu}$ coincide with the whole group G, and therefore.

$$G = A(g, L, \bar{\mu}) = B(g, R, \bar{\eta}) \ \ for\ each\ g \in G.$$

Thus, we have for $g \in G$,

$$\sup_{x \in A(g,L,\bar{\mu})} \bar{\eta}(x) = \sup_{x \in G} \bar{\eta}(x) = \bar{\eta}(e),$$

$$\sup_{y \in B(g,R,\bar{\eta})} \bar{\mu}(y) = \sup_{y \in G} \bar{\mu}(x) = \bar{\mu}(e)$$

Hence for each $g \in G$,

$$\bar{\eta} * \bar{\mu}(g) = \min\left(\bar{\eta}(e), \bar{\mu}(e)\right) = \min\left({}^{6}/_{7}, {}^{5}/_{7}\right) = {}^{5}/_{7}.$$

Therefore the free product of $\bar{\eta}$ and $\bar{\mu}$ is a constant fuzzy set in G.

Set protect: The set product of the fuzzy subgroups $\bar{\eta}$ and $\bar{\mu}$ is the fuzzy set $\bar{\eta} \circ \bar{\mu}$ in G, defined by

$\bar{\eta} \circ \bar{\mu}(\{e, (12)\}) = \left\{\frac{5}{7}\right\}$,

$$\bar{\eta} \circ \bar{\mu}(\{(34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}) = \left\{\frac{3}{7}\right\},$$

$\bar{\eta} \circ \bar{\mu}(\alpha) = \frac{2}{7}$ for any other permutation $\alpha\ in\ G$.

Clearly, $\bar{\eta} \circ \bar{\mu}$ is a fuzzy subgroup of G. Moreover, we have

$\bar{\mu} \leq \bar{\eta} \circ \bar{\mu} \ \ and\ \ \bar{\eta} \leq \bar{\eta} \circ \bar{\mu}.$

Proposition 1.1: Let η and μ be fuzzy subset of group G such that their respective supports A and B are subgroups of G. Then

$\min\{\eta * \mu(x), \eta * \mu(y)\} \leq \eta * \mu(xy)$, for all $x, y \in G$. Equivalently,

$$(\eta * \mu) \circ (\eta * \mu) \leq \eta * \mu.$$

Proposition 1.2 If η and μ are symmetric fuzzy sets in the group G, then

$\eta * \mu(g^{-1}) = \mu * \eta(g)$, for all $g \in G$.

Theorem 1.2: Let η and μ be fuzzy subgroups of the group G. Then $\eta * \mu$ is a fuzzy subgroup of G if and only if $\eta * \mu = \mu * \eta$.

Proof. Firstly, suppose that $\eta * \mu$ is a fuzzy subgroup of G. let $g \in G$. By Proposition 1.1, we get $\mu * \eta(g) = \eta * \mu(g^{-1}) = \eta * \mu(g)$.

Conversely, let $\eta * \mu = \mu * \eta$. If $x, y \in G$, then by Proposition 1.2, we have

$$\min\{\eta * \mu(x), \mu * \eta(y)\} \leq \eta * \mu(xy),$$

Also, by Proposition 1.2, we get $\eta * \mu(g^{-1}) = \eta * \mu(g) = \mu * \eta(g)$. Consequently, $\eta * \mu$ is a fuzzy subgroup of G.

The following example demonstrates that if η and μ are fuzzy subgroups of the group G and η is fuzzy normal in G, then the fuzzy subgroup $\eta * \mu$ of G may not be fuzzy normal in G.

Example1.2: Let us define the fuzzy subgroups η and μ of S_4 as follows. Let

$$\eta\big((1)\big) = \frac{6}{7}, \eta(\{(12)(34),(13)(24),(14)(23)\}) = \{\frac{4}{7}\}$$

$\eta(\alpha) = \frac{2}{7}$ for each 3-cycle α, and $\eta(\lambda) = 0$ for any other permutation λ in S_4.

Let $\mu(\{(1),(12)\}) = \left\{\frac{5}{7}\right\}, \mu(\{(12)(34),(34)\}) = \{\frac{3}{7}\}$,

$$\mu(\{(13)(24),(14)(23),(1324),(1423)\}) = \left\{\frac{1}{7}\right\},$$

And $\mu(\beta) = 0$ for any other permutation $\beta\ in\ S_4$. Clearly, η is fyzzy invariant in S_4.

Since $\mu\big((14)(124)\big) = \mu\big((24)\big) = 0 \neq \frac{5}{7} = \mu\big((12)\big) = \mu\big((124)(14)\big)$,

μ is not fuzzy invariant in S_4. The fuzzy subgroup $\eta * \mu$ of S_4 is not fuzzy invariant in S_4, because $\eta * \mu\big((14)(124)\big) = \frac{2}{7} \neq \frac{5}{7} = \eta * \mu\big((124)(14)\big)$.

Proposition 1.3. Let η and μ be fuzzy subgroups of the group G with supports A and B respectively.

i. If $\eta(e) \leq \mu(e)$, then $\eta * \mu(a) = \gamma(a) \leq \eta(e) \leq \mu(e) = \delta(a)$, for all $a \in A$.

ii. If $\mu(e) \leq \eta(e)$, then $\eta * \mu(b) = \delta(b) \leq \mu(e) \leq \eta(e) = \gamma(b)$, for all $b \in B$.

iii. If $\eta(e) = \mu(e)$, then $\eta * \mu(t) = \eta(e) = \mu(e)$, for all $t \in A \cap B$.

Proposition 1.4 Let G be a group and $\mu \in F(G)$. Then μ_t (level subset of G) is a subgroup of $G\ \forall\ t \in [0, \mu(e)]$, where e is the identity of G.

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