Some New Sufficient Conditions for The Oscillation of Nonlinear Ordinary Differential Equations

Jaffalah J. Amhalhil*

Department of Mathematics, Faculty of Education, Sirte University, Libya

*jaffalah.ali@su.edu.ly

Abstract: In this research, we shall discuss the oscillation behavior of solutions for second order nonlinear differential equations. By employing a generalized Riccati transformation and an averaging technique, we derive some new oscillation conditions for all solutions. Our results are extention for some well known oscillation results. The relevance of our results is illustrated by some examples.

Keywords. Sufficient Conditions, Oscillation, Nonlinear Differential Equations.

1. Introduction:

This paper is concerned with the problem of the oscillation of solutions for the following second order nonlinear ordinary differential equation:

$$\left(r(t)\psi(x(t))\dot{x}(t)\right)^{\bullet} + G(t, x(t)) = H(t, \dot{x}(t), x(t)) \quad , \ t \geq t_0 \tag{1}$$

Where

$r \in C(I, \Re_+)$, $\psi \in C(\Re, \Re_+)$, $I = [t_0, \infty), \Re = (-\infty, \infty), \Re_+ = (0, \infty)$

Throughout this paper, we shall impose the following conditions :

O_1: $G \in C(I \times \Re, \Re)$ and there exists $q \in C(I, \Re)$ such that $\frac{G(t, x(t))}{g(x(t))} \geq q(t)$ and $g \in C(\Re, \Re)$ such that $xg(x) > 0$ and $g'(x) \geq k > 0 \quad for \quad x \neq 0$.

O_2: $H \in C(I \times \Re^2, \Re)$ and there exists $m \in C(I, \Re_+)$ such that $\frac{H(t, \dot{x}(t), x(t))}{g(x(t))} \leq m(t) \quad for \quad x \neq 0$.

In the paper of Amhalhil [2], the oscillatry behavior of Equation (1) was discussed in the case when the right hand side equal to zero; that is when $H(t, \dot{x}(t), x(t)) = 0$. The oscillation property has wide applications especially physical sciences, technology and chemical reaction, one can see Agarwal et al. [1], Manojlovic [8] and Tiryaki et al. [10]. Howefere, among numerous papers dealing with this property we choose to refer to Elabbasy [3], Grace et al. [5], and the recent paper of Elabbasy et al. [4] and Salhin [9].

Here, in this paper, it will be assumed that the solutions of Equation (1) exist for $t_0 \geq 0$. Further, a solution $x(t)$ of Equation (1) is called regular (infinity continuable) if $x(t)$ exists for all $t \geq t_0$. A regular solution of Equation (1) is called oscillatory if it has arbitrarily large zeros and non oscillatory if it is eventually positive or eventually negative.

2. Main Results:

Theorem 2.1: Suppose that

$$O_3: \ r(t) \leq l_1 \qquad \forall t \geq t_0,$$

$$O_4: \ \frac{1}{\psi(x)} \geq l_2 \qquad \text{for all } x \in \Re,$$

$$O_5: \ \lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \int_{t_0}^{s} [q(u) - m(u)] du ds = \infty.$$

Then every solution of Equation (1) is oscillatory.

Proof. Without loss of generality, we may assume that there exists a solution $x(t) > 0$ on $[t_0, \infty)$ for some $T_1 \geq t_0 > 0$. Define

$$\omega(t) = \frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))}, \qquad t \geq T_1.$$

(2)

Then, for $t \geq T_1$, we get

$$\dot{\omega}(t) = \frac{(r(t)\psi(x(t))\dot{x}(t))^{\bullet}}{g(x(t))} - \frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))},$$

And so, from Equation (1), we have

$$\dot{\omega}(t) = \frac{H(t,\dot{x}(t),x(t))}{g(x(t))} - \frac{G(t,x(t))}{g(x(t))} - \frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))},$$

Then, for every $t \geq T_1$, we obtain

$$\dot{\omega}(t) \leq m(t) - q(t) - \frac{r(t)\psi(x(t))\dot{x}^2(t)g'(x(t))}{g^2(x(t))},$$

(3)

Thus, for every $t \geq T_1$, we have

$$\omega(t) \leq \omega(T_1) - \int_{T_1}^{t} \frac{r(s)\psi(x(s))\dot{x}^2(s)g'(x(s))}{g^2(x(s))}\,ds - \int_{T_1}^{t}[q(s) - m(s)]ds.$$

Then, by definition of ω, we get

$$\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \leq C - \int_{T_1}^{t} \frac{r(s)\psi(x(s))\dot{x}^2(s)g'(x(s))}{g^2(x(s))}\,ds - \int_{T_1}^{t}[q(s) - m(s)]ds, \qquad t \geq T_1,$$

Where $C = \omega(T_1)$ is a real constant. Now, from conditions O_3 and O_4, we have

$$\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \leq C - \frac{kl_2}{l_1}\int_{T_1}^{t}\left[\frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}\right]^2 ds - \int_{T_1}^{t}[q(s) - m(s)]ds,$$

Integrating again and dividing by t, we obtain

$$\frac{1}{t}\int_{T_1}^{t}\frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}ds \le C\left(1-\frac{T_1}{t}\right)-\frac{d}{t}\int_{T_1}^{t}\int_{T_1}^{s}\left[\frac{r(u)\psi(x(u))\dot{x}(u)}{g(x(u))}\right]^2 duds$$
$$-\frac{1}{t}\int_{T_1}^{t}\int_{T_1}^{s}[q(u)-m(u)]duds, \qquad t\ge T_1$$

where $d=\frac{kl_2}{l_1}$ is a positive constant. Then, for $t\ge T_1$, by condition O_5, we obtain

$$\lim_{t\to\infty}\frac{1}{t}\int_{T_1}^{t}\frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}ds=-\infty$$

(4)

Now, defining

$$V(t)=\left|\int_{T_1}^{t}\frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}ds\right|,$$

(5)

and applying Schwarz's inequality, we have

$$V^2(t)=\left|\int_{T_1}^{t}\frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}ds\right|^2 \le \int_{T_1}^{t}\left|\frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}\right|^2 ds * \int_{T_1}^{t}|1|^2 ds$$
$$\le (t-T_1)\int_{T_1}^{t}\left|\frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}\right|^2 ds , \quad t\ge T_1.$$

Thus,

$$V^2(t) \le t\int_{T_1}^{t}\left|\frac{r(s)\psi(x(s))\dot{x}(s)}{g(x(s))}\right|^2 ds.$$

(6)

Using condition O_5 we obtain

$$\frac{-V(t)}{t}+\frac{d}{t}\int_{T_2}^{t}\int_{T_2}^{s}\left[\frac{r(u)\psi(x(u))\dot{x}(u)}{g(x(u))}\right]^2 duds \le 0.$$

Then, for all $t \ge T_2$, we have

$$\frac{-V(t)}{t}+\frac{d}{t}\int_{T_2}^{t}\frac{V^2(s)}{s}ds \le 0,$$

which follows that

$$\frac{d}{t}\int_{T_2}^{t}\frac{V^2(s)}{s}ds \le \frac{V(t)}{t}.$$

(7)

Thus, for all $t \ge T_2$, inequality (7) becomes

$$\frac{d^2}{t^2}\left[\int_{T_2}^{t}\frac{V^2(s)}{s}ds\right]^2 \le \frac{V^2(t)}{t^2}.$$

Then, for $t \ge T_2$, we define

$$\varphi(t)=\int_{T_2}^{t}\frac{V^2(s)}{s}ds.$$

(8)

Then, we get

$$\frac{d^2}{t} \leq \frac{\varphi'(t)}{\varphi^2(t)}.$$

Integrating the last inequality from T_2 to t, we obtain

$$d^2 \ln\left(\frac{t}{T_2}\right) \leq \frac{1}{\varphi(T_2)} - \frac{1}{\varphi(t)} \leq \frac{1}{\varphi(T_2)}. \quad (9)$$

This is a contradiction and hence the proof is complete.

Example 2.1: Consider the equation:

$$\left[\left(\frac{1}{t+1}\right)\left(\frac{x^2(t)+1}{x^2(t)+2}\right)\dot{x}(t)\right]^{\bullet} + x(t)\left(t^2 + x^2(t)\right) = \frac{x^3(t)\dot{x}^2(t)}{t^3(\dot{x}^2(t)+1)(x^2(t)+1)}, \quad t > 0. \quad (10)$$

We note that

$$0 < r(t) = \frac{1}{t+1} < 1 \qquad for\ all\ t > 0,$$

$$\psi(x) = \frac{x^2+1}{x^2+2} > 0 \qquad and \qquad \frac{1}{\psi(x)} = \frac{x^2+2}{x^2+1} > 1 \qquad \forall x \in \Re,$$

$$\frac{G(t,x(t))}{g(x(t))} = \frac{x(t)\left(t^2 + x^2(t)\right)}{x(t)} = \left(t^2 + x^2(t)\right) \geq t^2 = q(t) \qquad for\ x \neq 0 \quad and \quad t \in [t_0, \infty)$$

$$and \quad xg(x) = x^2 > 0 \quad and \quad g'(x) = 1 > 0 \quad for \quad x \neq 0,$$

$$\frac{H(t,\dot{x}(t),x(t))}{g(x(t))} = \frac{x^3(t)\dot{x}^2(t)}{x(t)t^3\left(\dot{x}^2+1\right)\left(x^2+1\right)} \leq \frac{1}{t^3} = m(t) \qquad \forall x \neq 0,\ \dot{x} \in \Re, t > 0,$$

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}[q(u) - m(u)]duds = \lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\int_{t_0}^{s}\left[u^2 - \frac{1}{u^3}\right]duds$$

$$=\lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t}\left[\frac{s^3}{3}-\frac{1}{2s^2}-\frac{t_0^3}{3}+\frac{1}{2t_0^2}\right]ds$$

$$=\lim_{t\to\infty}\frac{1}{t}\left[\frac{s^4}{12}+\frac{1}{2s}-\frac{t_0^3 s}{3}+\frac{s}{2t_0^2}\right]_{t_0}^{t}=\infty,$$

It follows that by Theorem 2.1, Equation (10) is oscillatory.

Theorem 2.2: Suppose that the condition O_4 holds and

$$O_6: \quad \int_{t_0}^{\infty}\frac{ds}{r(s)}=\infty,$$

$$O_7: \int_{t_0}^{\infty}[q(s)-m(s)]ds=\infty,$$

Then every solution of Equation (1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of Equation (1), say $x(t)>0 \quad for\ t\ge T_0\ge t_0$. Define

$$\omega(t)=\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))},$$

Then, for $t\ge T_0$, we get

$$\dot{\omega}(t)\le m(t)-q(t),$$

Then, for every $t\ge T_0$, we obtain

$$\omega(t)\le\omega(T_0)-\int_{T_0}^{t}[q(s)-m(s)]ds. \quad (11)$$

Then, by definition of ω, we get

$$\frac{r(t)\psi(x(t))\dot{x}(t)}{g(x(t))} \le \omega(T_0) - \int_{T_0}^{t}[q(s)-m(s)]ds.$$

By condition O_7 there exists $T_1 \ge T_0$, such that

$$\dot{x}(t) < 0, \qquad for \ t \ge T_1,$$

Also, by condition O_7 implies that there exists $T_2 \ge T_1$, such that

$$\int_{T_1}^{T_2}[q(s)-m(s)]ds = 0 \qquad \text{and} \qquad \int_{T_2}^{t}[q)s)-m(s)]ds \ge 0, \qquad t \ge T_2.$$

Integrating Equation (1), we get

$$r(t)\psi(x(t))\dot{x}(t) = r(T_2)\psi(x(T_2))\dot{x}(T_2) + \int_{T_2}^{t} H(s,\dot{x}(s),x(s))ds - \int_{T_2}^{t} G(s,x(s))ds.$$

Then, for $t \ge T_2$, we have

$$r(t)\psi(x(t))\dot{x}(t) \le r(T_2)\psi(x(T_2))\dot{x}(T_2) - \int_{T_2}^{t} g(x(s))[q(s)-m(s)]ds$$

$$\le r(T_2)\psi(x(T_2))\dot{x}(T_2) - g(x(t))\int_{T_2}^{t}[q(s)-m(s)]ds + \int_{T_2}^{t}\dot{x}(s)g'(x(s))\int_{T_2}^{s}[q(u)-m(u)]duds.$$

Hence, for $t \ge T_2$, we obtain

$$r(t)\psi(x(t))\dot{x}(t) \le r(T_2)\psi(x(T_2))\dot{x}(T_2).$$

Then, for every $t \ge T_2$, we have

$$r(t)\dot{x}(t) \le r(T_2)\psi(x(T_2))\dot{x}(T_2).l_2 , \qquad t \ge T_2.$$

Thus,

$$\dot{x}(t) \le l_2\, r(T_2)\psi(x(T_2))\dot{x}(T_2)\frac{1}{r(t)}.$$

(12)

Integrating the inequality (12) from T_2 *to* t, we obtain

$$x(t) \le x(T_2) + l_2 r(T_2)\psi(x(T_2)\dot{x}(T_2))\int_{T_2}^{t}\frac{ds}{r(s)} \to -\infty \qquad as \quad t \to \infty,$$

which is a contradiction to the fact that $x(t) > 0$; hence, the proof is complete.

Example 2.2: Consider the equation:

$$\left[\left(\frac{t^2}{t^2+1}\right)\left(\frac{2x^2(t)+3}{2x^2(t)+4}\right)\dot{x}(t)\right]^{\bullet} + x(t)\left(\frac{1}{t}+x^4(t)\right) = \frac{x^3(t)\dot{x}^4(t)(\sin t - 1)}{\left(1+x^2(t)\right)\left(\dot{x}^4(t)+1\right)}, \quad t > 0.$$

(13)

We note that

$$0 < r(t) = \frac{t^2}{t^2+1} < 1 \qquad and \qquad \int_{t_0}^{\infty}\frac{ds}{r(s)} = \int_{t_0}^{\infty}\left(1+\frac{1}{s^2}\right)ds = \infty, \quad t > 0,$$

$$\psi(x)=\frac{2x^2+3}{2x^2+4}>0 \quad and \quad \frac{1}{\psi(x)}=\frac{2x^2+4}{2x^2+3}>1 \quad for\ all\ x\in\Re,$$

$$\frac{G(t,x(t))}{g(x(t))}=\frac{x(t)\left(\frac{1}{t}+x^4(t)\right)}{x(t)}=\left(\frac{1}{t}+x^4(t)\right)\geq\frac{1}{t}=q(t) \quad for\ all\ x\neq 0\ and\ t\in[t_0,$$

$$and\ \ xg(x)=x^2>0 \quad and \quad g'(x)=1>0 \quad for\ x\neq 0,$$

$$\frac{H(t,\dot{x}(t),x(t))}{g(x(t))}=\frac{x^3\dot{x}^4(\sin t-1)}{(1+x^2)(\dot{x}^4+1)}\leq(\sin t-1)=m(t) \quad \forall x,\dot{x}\in\Re,\ and\ t\in[t_0,\infty)$$

$$\int_{t_0}^{\infty}[q(s)-m(s)]ds=\int_{t_0}^{\infty}\left[\frac{1}{s}-\sin s+1\right]ds=\ln s+\cos s+s\Big|_{t_0}^{\infty}=\infty,$$

It follows that by Theorem 2.2, Equation (13) is oscillatory.

Remark 2.1: Theorem 2.1 and Theorem 2.2 extend the results of Graef et. al. [6] and Li and Cheng [7].

3. References.

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بعض الشروط الكافية الجديدة لتذبذب المعادلات التفاضلية العادية الغير خطية

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