Properties for Certain Subclasses of Meromorphic Functions Associated with Differential Operator

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Abstract

In this paper, we introduce and study the classes $\sum S^*(k,\alpha,\beta)$ and $\sum C^*(k,\alpha,\beta)$ of meromorphic univalent functions in $\mathbb{U}^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$ and investiage distortion theorems, convex linear combinations and integral transforms of functions belonging to these classes.

الملخص

في هذا البحث عرفنا الفصول $\sum S^*(k,\alpha,\beta)$ و $\sum C^*(k,\alpha,\beta)$ للدوال الميروMورفية أحادية التكافؤ النجمية والمحدبة على الترتيب والمعرفة على قرص الوحدة المثقوب $\mathbb{U}^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$. ودرسنا بعض الخصائص لهذه الفصول وهي التشوه – التركيبات الخطية المحدبة– التحويلات التكاملية - حاصل ضرب هادمرد (الالتفاف)، وكل النتائج التي حصلنا عليها قاطعة.

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Introduction.

Let $\sum$ denote the class of meromorphic functions of the form:

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad (a_n \geq 0) \qquad (1.1)$$

which are analytic and univalent in the punctured unit disc $\mathbb{U}^* = \{z \in \mathbb{C}: 0 < |z| < 1\} =$

$\mathbb{U}\backslash\{0\}$. A function $f(x) \in \Sigma$ is said to be meromorphically starlike of order $\alpha(0 \leq \alpha < 1$ if and only if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha;\ (z \in \mathbb{U}^*) \qquad (1.2)$$

and the class of such functions is denoted by $\Sigma^*(\alpha)$ A function $f(x) \in \Sigma$ is said to be meromorphically convex of order $\alpha(0 \leq \alpha < 1)$ if and only if

$$-Re\left\{1+\frac{zf'(z)}{f(z)}\right\} > \alpha;\ (z \in \mathbb{U}^*) \qquad (1.3)$$

and the class of such functions is denoted by $\Sigma_k^*(\alpha)$ The classes $\Sigma^*(\alpha)$ and $\Sigma_k^*(\alpha)$, were introduced and studied by Pommerenke [9], Miller [7], Mogra et al. [8], Cho [3], Cho et al. [4], and Aouf ([1] and [2]).

For a function $f(z)$ defined by (1.1) let

$$I^0 f(z) = f(z)$$

$$I^1 f(z) = zf'(z) + \tfrac{2}{z}$$

$$I^2 f(z) = z\big(I^1 f(z)\big)' + \tfrac{2}{z}$$

and for $k \in \mathbb{N} = \{1,2,3,\dots\}$

$$I^k f(z) = z\left(I^{k-1} f(z)\right)' + \tfrac{2}{z}$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} n^k a_n z^n.$$

The operator I^k was introduced by Frasin and Darus [6].

With the help of the differential operator I^k, we define the classes $\Sigma S^*(k,\alpha,\beta)$ and $\Sigma C^*(k,\alpha,\beta)$ as follows:

Denote by $\Sigma S^*(k,\alpha,\beta)$ the class of functions $f(z)$ of the form (1.1) which satisfy

$$\left|\frac{\frac{z(I^k f(z))'}{I^k f(z)} + 1}{\frac{z(I^k f(z))'}{I^k f(z)} + \alpha - 1}\right| < \beta(0 \leq \alpha < 1, 0 < \beta \leq 1), k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \mathbb{U}^*. \qquad (1.4)$$

Let $\sum C^*(k,\alpha,\beta)$ be the class of functions $f(z)$ of the form for which $zf'(z) \in \sum S^*(k,\alpha,\beta)$

We note that for different choices of k,α,β we obtain many classes studied earlier (see [5, with $a_0 = 1$] and [8]).

To prove our main result in this paper, we need the following lemmas given by El-Ashwah and Aouf [5,with $a_0 = 1$].

Lemma 1. *The function* $f(z) \in \sum S^*(k,\alpha,\beta)$ *if and only if*

$$\sum_{n=1}^{\infty} n^k[(1+\beta)n + (2\alpha-1)\beta]a_n \le 2\beta(1-\alpha) \qquad (1.5)$$

Lemma 2. *The function* $f(z) \in \sum C^*(k,\alpha,\beta)$ *if and only if*

$$\sum_{n=1}^{\infty} n^{k+1}[(1+\beta)n + (2\alpha-1)\beta]a_n \le 2\beta(1-\alpha) \qquad (1.6)$$

Remark1. *(i) Putting* $k = 0$ *in Lemma 1, we obtain the result obtained by Mogra et al. [8, Thoerem 1].*

(ii) Putting $k = 0$ *in Lemma 2, we obtain the result obtained by Mogra et al. [8, Thoerem 2].*

Distortion theorems.

Theorem 1. *Let the function* $f(z) \in \sum S^*(k,\alpha,\beta)$. *Then for* $|z| = r < 1$ *we have*

$$\frac{1}{r} - \frac{\beta(1-\alpha)}{1+\alpha\beta} r < |f(z)| \le \frac{1}{r} + \frac{\beta(1-\alpha)}{1+\alpha\beta} r \qquad (1.2)$$

The result is sharp for the function $f(z)$ *given by*

$$f(z) = \frac{1}{z} + \frac{\beta(1-\alpha)}{1+\alpha\beta} z \ at\ z = ir, r. \qquad (2.2)$$

Proof. Suppose $f(z)$ is in the class $\sum S^*(k,\alpha,\beta)$. In view of Lemma1, we have

$$2(1+\alpha\beta) \le \sum_{n=1}^{\infty} n^k[(1+\beta)n + (2\alpha-1)\beta]a_n \le 2\beta(1-\alpha)$$

that is, that

$$a_n \leq \frac{\beta(1-\alpha)}{1+\alpha\beta} \qquad (2.3)$$

Thus for $0 < |z| = r < 1$, from (1.1) and (2.3) we have

$$|f(z)| = \left|\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \right| \leq \frac{1}{|z|} + \sum_{n=1}^{\infty} a_n |z|^n \leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n \leq \frac{1}{r} + \frac{\beta(1-\alpha)}{1+\alpha\beta} r\,, \qquad (2.4)$$

and

$$|f(z)| \geq \frac{1}{|z|} - \sum_{n=1}^{\infty} a_n \geq \frac{1}{r} - r\sum_{n=1}^{\infty} a_n \geq \frac{1}{r} - \frac{\beta(1-\alpha)}{1+\alpha\beta} r. \qquad (2.5)$$

It can easily seen that the function $f_1(z)$ defined by (2.2) is the extrenal for the theorem.

Theorem 2. *Let the function* $f(z) \in \sum S^*(k,\alpha,\beta)$. *Then for* $|z| = r < 1$, *we have*

$$\frac{1}{r^2} - \frac{\beta(1-\alpha)}{1+\alpha\beta} r < |f'(z)| \leq \frac{1}{r^2} + \frac{\beta(1-\alpha)}{1+\alpha\beta} r \qquad (2.6)$$

Sharpness holds for function $f(z)$ *given by (2.2).*

Proof. From Lemma1. and (2.3), we have

$$\sum_{n=1}^{\infty} n a_n \leq \frac{\beta(1-\alpha)}{1+\alpha\beta} \qquad (2.7)$$

Since the remaining part of the proof is similar to the proof of Theorem 1, we omit the details.

Theorem 3. *Let* $f(z) \in \sum S^*(k,\alpha,\beta)$, *then* $f(z)$ *is meromorphically convex*

$$r = \inf_n \left\{\frac{n^{k-1}(1-\delta)[(1+\beta)n + (2\alpha-1)\beta + 1]}{2\beta n(1-\alpha)(n+2-\beta)}\right\}^{\frac{1}{n+1}} \quad (n \in \mathbb{N}). \qquad (2.8)$$

Sharpness holds for

$$f_n(z) = \frac{1}{z} + \frac{2\beta(1-\alpha)}{n^k[(1+\beta)n + (2\alpha-1)\beta]} z^n \quad (n \geq 1)\,. \qquad (2.9)$$

Proof. It is sufficient to show for $f(z) \in \sum S^*(k, \alpha, \beta)$, that

$$\left|\frac{zf''(z)}{f'(z)} + 2\right| \leq 1 - \delta, |z| < r(k, \alpha, \beta, \delta),$$

where $r(k, \alpha, \beta, \delta)$ is the largest value of r for which the inequality (2.8) holds true.

For $f(z)$ of the form (1.1), we have

$$\left|\frac{zf''(z)}{f'(z)} + 2\right| \leq \frac{\sum_{n=1}^{\infty} n(n+1)a_n r^{n+1}}{1 - \sum_{n=1}^{\infty} n a_n r^{n+1}}$$

Thus

$$\left|\frac{zf''(z)}{f'(z)} + 2\right| \leq 1 - \delta |z| < r; 0 \leq \delta < 1$$

if and only if

$$\sum_{n=1}^{\infty} \frac{n(n+2-\delta)}{(1-\delta)} a_n r^{n+1} \leq 1 \; (0 \leq \delta < 1) \qquad (2.10)$$

But, by Lemma1 , (2.10) will be true if

$$\left(\frac{n(n+2-\delta)}{1-\delta}\right) r^{n+1} \leq \frac{n^k[(1+\beta)n + (2\alpha-1)\beta + 1]}{2\beta(1-\alpha)}$$

that is, if

$$r \leq \left\{\frac{n^{k-1}(1-\delta)(1+\beta)n + (2\alpha-1)\beta + 1}{2\beta n(1-\alpha)(n+2-\beta)}\right\}^{\frac{1}{n+1}} \quad (n \in \mathbb{N}). \qquad (2.11)$$

Theorem follows easily from (2.11).

Convex linear combinations.

In this section we shall prove that the classes $\sum S^*(k,\alpha,\beta)$ and $\sum C^*(k,\alpha,\beta)$ are closed under convex linear combinations.

Theorem 4. *Let*

$$f_0(z) = \frac{1}{z} \tag{3.1}$$

and $f_n(z)(n \geq 1)$ *be given by (2.9). Then* $f(z) \in \sum S^*(k,\alpha,\beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty} \eta_n f_n(z) \quad where\ \eta_n \geq 0\ and \sum_{n=0}^{\infty} \eta_n = 1. \tag{3.2}$$

Proof. Assume that

$$f(z) = \sum_{n=0}^{\infty} \eta_n f_n(z) \quad with\ \ \eta_n \geq 0\ and \sum_{n=0}^{\infty} \eta_n = 1.$$

$$f(z) = \sum_{n=0}^{\infty} \eta_n f_n(z) = \eta_0 f_0(z) + \sum_{n=1}^{\infty} \eta_n f_n(z)$$

$$= (1 - \sum_{n=1}^{\infty} \eta_n\ ) f_0(z) + \sum_{n=1}^{\infty} \eta_n f_n(z)$$

$$= (1 - \sum_{n=1}^{\infty} \eta_n\ )\frac{1}{z} + \sum_{n=1}^{\infty} \eta_n \left(\frac{1}{z} + \frac{2\beta(1-\alpha)}{n^k[(1+\beta)n + (2\alpha-1)\beta]} z^n\right)$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2\beta(1-\alpha)}{n^k[(1+\beta)n + (2\alpha-1)\beta]} \eta_n z^n.$$

Then it follows that

$$\sum_{n=1}^{\infty} \frac{n^k[(1+\beta)n + (2\alpha-1)\beta])}{2\beta(1-\alpha)} \eta_n \frac{2\beta(1-\alpha)}{n^k[(1+\beta)n + (2\alpha-1)\beta]}$$

$$\sum_{n=1}^{\infty} \eta_n = 1 - \eta_0 \leq 1.$$

So, by Lemma1, $f(z) \in \sum S^*(k, \alpha, \beta)$.

Conversely, suppose $f(z) \in \sum S^*(k, \alpha, \beta)$, then

$$a_n \leq \frac{2\beta(1-\alpha)}{n^k[(1+\beta)n + (2\alpha - 1)\beta]} \ (n = 1,2,3, \dots),$$

setting

$$\eta_n = \frac{n^k[(1+\beta)n + (2\alpha - 1)\beta])}{2\beta(1-\alpha)} a_n, n = 1,2,3, \dots and\ \eta_0 = 1 - \sum_{n=1}^{\infty} \eta_n,$$

it follows that

$$f(z) = \sum_{n=0}^{\infty} \eta_n f_n(z).$$

This completes the proof of Theorem4.

Theorem 5. *Let* $f_0(z)$ *given by (3.1) and* $f_n(z)$ *be given by*

$$f_n(z) = \frac{1}{z} + \frac{2\beta(1-\alpha)}{n^{k+1}[(1+\beta)n + (2\alpha - 1)\beta]} z^n \quad (n \geq 1). \tag{3.3}$$

Then $f(z) \in \sum C^*(k, \alpha, \beta)$ *if and only if it can be expressed in the form (3.2).*

Theorem 6. *The class* $\sum S^*(k, \alpha, \beta)$ *is closed under convex linear combination.*

Proof. Suppose that

$$f_n(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n \ (a_{n,j} \geq 0, j = 1, 2, 3, \dots), \tag{3.4}$$

are in the class $f(z) \in \sum S^*(k,\alpha,\beta)$. Let

$$f(z) = (1-s)f_1(z) + sf_2(z), \qquad 0 \le s \le 1$$

Then

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} [(1-s)a_{n,1} + sa_{n,2}]z^n$$

In view of Lemma1, we have

$$\sum_{n=1}^{\infty} n^k[(1+\beta)n + (2\alpha-1)\beta][(1-s)a_{n,1} + sa_{n,2}]$$

$$= (1-s)\sum_{n=1}^{\infty} n^k[(1+\beta)n + (2\alpha-1)\beta]a_{n,1} + s\sum_{n=1}^{\infty} n^k[(1+\beta)n + (2\alpha-1)\beta]a_{n,2}$$

$$\le 2\beta(1-s)(1-\alpha) + 2\beta s(1-\alpha) = 2\beta(1-\alpha).$$

This shows that $f(z) \in \sum S^*(k,\alpha,\beta)$. and hence the proof of Theorem is completed.

Theorem 7. *The class* $\sum C^*(k,\alpha,\beta)$ *is closed under convex linear combination.*

Remark 2. *Putting* $k = 0$ *in Theorem 4, we obtain the result obtained by Mogra et al. [8, Thoerem 5].*

Integral transforms

Theorem 8. *If* $\sum S^*(k,\alpha,\beta)$ *then the integral transforms*

$$F_c(z) = c\int_0^1 u^c f(uz)du, \qquad (c > 0) \qquad (4.1)$$

are in the class $\Sigma S^*(\gamma)$ *where*

$$\gamma = \gamma(k, \alpha, \beta, c) = \frac{(1+\alpha\beta)(2+c) - c\beta(1-\alpha)}{(1+\alpha\beta)(2+c) + c\beta(1-\alpha)} \qquad (4.2)$$

The result is best possible for the function

$$f(z) = \frac{1}{z} + \frac{\beta(1-\alpha)}{1+\alpha\beta} z \qquad (4.3)$$

Proof. Suppose that $\Sigma S^*(k, \alpha, \beta)$ then we have

$$F_c(z) = c\int_0^1 u^c f(uz)du = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{c\, a_n}{c+n+1} z^n.$$

To prove that $F_c(z)$ is meromorphically starlike function of order γ, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{(n+\gamma)}{1-\gamma} \cdot \frac{c\, a_n}{c+n+1} \leq 1. \qquad (4.4)$$

Since $f(z) \in \Sigma S^*(k, \alpha, \beta)$ then

$$\sum_{n=1}^{\infty} \frac{n^k[(1+\beta)n + (2\alpha-1)\beta]}{2\beta(1-\alpha)} a_n \leq 1 \qquad (4.5)$$

Thus (4.4) will be satisfied if

$$\frac{(n+\gamma)}{(1-\gamma)(c+n+1)} \leq \frac{n^k[(1+\beta)n + (2\alpha-1)\beta]}{2\beta(1-\alpha)} \quad for\ each\ n,$$

or

$$\gamma \leq \frac{n^k[(1+\beta)n + (2\alpha-1)\beta](c+n+1) - 2\beta(1-\alpha)cn}{n^k[(1+\beta)n + (2\alpha-1)\beta](c+n+1) + 2\beta(1-\alpha)cn}. \qquad (4.6)$$

Since the right hand side of (4.6) is an increasing function of n, putting n =1 in (4.6) we get

$$\gamma \leq \frac{[(1+\beta)+(2\alpha-1)\beta](2+c)-2c\beta(1-\alpha)}{[(1+\beta)+(2\alpha-1)\beta](2+c)+2c\beta(1-\alpha)}$$

and hence the proof of Theorem is completed.

Similarly we can find the integral transforms for the class $f(z) \in \sum C^*(k,\alpha,\beta)$.

Remark 1. *It is interesting to note that for* $c = 1$ *and* $(\alpha,\beta) = (0,1)$. *Theorem 8 gives that if* $f(z) \in \sum S^*(k,\alpha,\beta)$ *then*

$$F_1(z) = c\int_0^1 uf(uz)du$$

Hadamard products

Let the functions be defined by (3.4), then the Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by $f_j(z)$

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1}a_{n,2}z^n = (f_2 * f_1)(z). \qquad (5.1)$$

We prove the following results for functions in the classes $\sum S^*(k,\alpha,\beta)$ and $\sum C^*(k,\alpha,\beta)$

Theorem 9. *Let the functions* $f_j(z)(j = 1,2)$ *defined by (3.4) be in the class* $\sum S^*(k,\alpha,\beta)$. *Then*$(f_1 * f_2)(z) \in \sum S^*(k,\varphi,\beta)$, *where*

$$\varphi = 1 - \frac{\beta(1+\beta)(1-\alpha)^2}{(1+\alpha\beta)^2+\beta^2(1-\alpha)^2}. \qquad (5.2)$$

Sharpness holds for functions

$$f_j(z) = \frac{1}{z} + \frac{\beta(1-\alpha)}{1+\alpha\beta}z \quad (j = 1,2). \qquad (5.3)$$

Proof. Employing the technique used earlier by Schild and Silverman [10] for univalent functions, we need to find the largest real parameter φ such that

$$\sum_{n=1}^{\infty} \frac{n^k[(1+\beta)n+(2\varphi-1)\beta+1]}{2\beta(1-\varphi)}\, a_{n,1}a_{n,2} \ \leq 1. \qquad (5.4)$$

Since $(f_j)(z) \in \sum S^*(k,\alpha,\beta)\ (j=1,2),$ we readily see that

$$\sum_{n=1}^{\infty} \frac{n^k[(1+\beta)n+(2\alpha-1)\beta+1]}{2\beta(1-\alpha)}\, a_{n,1} \ \leq 1 \qquad (5.5)$$

and

$$\sum_{n=1}^{\infty} \frac{n^k[(1+\beta)n+(2\alpha-1)\beta+1]}{2\beta(1-\alpha)}\, a_{n,2} \ \leq 1. \qquad (5.6)$$

By Cauchy-Schwarz inequality we have

$$\sum_{n=1}^{\infty} \frac{n^k[(1+\beta)n+(2\alpha-1)\beta+1]}{2\beta(1-\alpha)}\, \sqrt{a_{n,1}a_{n,2}} \ \leq 1. \qquad (5.7)$$

Thus it is sufficient to show that

$$\frac{n^k[(1+\beta)n+(2\varphi-1)\beta+1]}{2\beta(1-\varphi)}\, a_{n,1}a_{n,2} \leq \frac{n^k[(1+\beta)n+(2\alpha-1)\beta+1]}{2\beta(1-\alpha)}\, \sqrt{a_{n,1}a_{n,2}} \qquad (5.8)$$

or, equivalently, that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{(1-\varphi)[(1+\beta)n+(2\alpha-1)\beta+1]}{(1-\alpha)[(1+\beta)n+(2\varphi-1)\beta+1]}. \qquad (5.9)$$

Hence, in the light of the inequality (5.7), it is sufficient to prove that

$$\frac{2\beta(1-\alpha)}{n^k[(1+\beta)n+(2\alpha-1)\beta+1]} \le \frac{(1-\varphi)[(1+\beta)n+(2\alpha-1)\beta+1]}{(1-\alpha)[(1+\beta)n+(2\varphi-1)\beta+1]}. \quad (5.10)$$

It follows from (5.10) that

$$\varphi \le 1-\frac{2\beta(1+n)(1+\beta)(1-\alpha)^2}{n^k[(1+\beta)n+(2\alpha-1)\beta+1]^2+4\beta^2(1-\alpha)^2}. \quad (5.11)$$

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Now defining the function $G(n)$ by

$$G(n)=1-\frac{2\beta(1+n)(1+\beta)(1-\alpha)^2}{n^k[(1+\beta)n+(2\alpha-1)\beta+1]^2+4\beta^2(1-\alpha)^2}. \quad (5.12)$$

We see that $G(n)$ is anincreasing function of $n(n \ge 1)$. Therefore, we conclude that

$$\varphi \le G(1)=1-\frac{\beta(1+\beta)(1-\alpha)^2}{(1+\alpha\beta)^2+\beta^2(1-\alpha)^2} \quad (5.13)$$

and hence the proof of Theorem 9 is completed.

Theorem 10. *Let the functions $(f_j)(z)(j=1,2)$ defined by (3.4) be in the class $\sum C^*(k,\alpha,\beta)$. Then $(f_1 * f_2)(z) \in \sum C^*(k,\varphi,\beta)$, where φ is given by (5.2). The result is sharp for functions $(f_j)(z)(j=1,2)$ given by (5.3).*

Theorem 11. *Let the function $(f_1)(z)$ defined by (3.4) be in the class $\sum S^*(k,\alpha,\beta)$. Suppose also that the function $(f_2)(z)$ defined by (3.4) be in the class $\sum S^*(k,\delta,\beta)$. Then $(f_1 * f_2)(z) \in \sum S^*(k,\rho,\beta)$, where*

$$\rho = 1-\frac{\beta(1+\beta)(1-\alpha)(1-\delta)}{(1+\alpha\beta)(1+\delta\beta)+\beta^2(1-\alpha)(1-\delta)} \quad (5.14)$$

Sharpness holds for functions

$$f_1(z)=\frac{1}{z}+\frac{\beta(1-\alpha)}{1+\alpha\beta}z \quad (5.15)$$

and

$$f_2(z) = \frac{1}{z} + \frac{\beta(1-\delta)}{1+\delta\beta} z. \qquad (5.16)$$

Proof. We need to find the largest real parameter ρ such that

$$\sum_{n=1}^{\infty} \frac{n^k[(1+\beta)n + (2\rho - 1)\beta + 1]}{2\beta(1-\rho)} a_{n,1}a_{n,2} \le 1. \qquad (5.17)$$

Since $f_1(z) \in \sum S^*(k,\alpha,\beta)$, we readily see that

$$\sum_{n=1}^{\infty} \frac{n^k[(1+\beta)n + (2\alpha - 1)\beta + 1]}{2\beta(1-\alpha)} a_{n,1} \le 1, \qquad (5.18)$$

and since $f_2(z) \in \sum S^*(k,\delta,\beta)$ we readily see that

$$\sum_{n=1}^{\infty} \frac{n^k[(1+\beta)n + (2\delta - 1)\beta + 1]}{2\beta(1-\delta)} a_{n,2} \le 1. \qquad (5.19)$$

By Cauchy-Schwarz inequality we have

$$\sum_{n=1}^{\infty} \frac{n^k[(1+\beta)n + (2\alpha - 1)\beta + 1]^{\frac{1}{2}}[(1+\beta)n + (2\delta - 1)\beta + 1]^{\frac{1}{2}}}{\sqrt{2\beta(1-\alpha)}\sqrt{2\beta(1-\delta}} \sqrt{a_{n,1}a_{n,2}} \le 1. \qquad (5.20)$$

Thus it is sufficient to show that

$$\frac{n^k[(1+\beta)n + (2\rho - 1)\beta + 1]}{2\beta(1-\rho)} a_{n,1}a_{n,2} \le \frac{n^k[(1+\beta)n + (2\alpha - 1)\beta + 1]^{\frac{1}{2}}[(1+\beta)n + (2\delta - 1)\beta + 1]^{\frac{1}{2}}}{\sqrt{2\beta(1-\alpha)}\sqrt{2\beta(1-\delta}} \sqrt{a_{n,1}a_{n,2}} \qquad (5.21)$$

or, equivalently, that

$$\sqrt{a_{n,1}a_{n,2}} \leq \frac{2\beta(1-\rho)[(1+\beta)n+(2\alpha-1)\beta+1]^{\frac{1}{2}}[(1+\beta)n+(2\delta-1)\beta+1]^{\frac{1}{2}}}{\sqrt{2\beta(1-\alpha)}\sqrt{2\beta(1-\delta}\,[(1+\beta)n+(2\rho-1)\beta+1]}.$$

Hence, in the light of the inequality (5.20), it is sufficient to prove that

$$\frac{\sqrt{2\beta(1-\alpha)}\sqrt{2\beta(1-\delta}}{n^k[(1+\beta)n+(2\alpha-1)\beta+1]^{\frac{1}{2}}[(1+\beta)n+(2\delta-1)\beta+1]^{\frac{1}{2}}} \leq \frac{2\beta(1-\rho)[(1+\beta)n+(2\alpha-1)\beta+1]^{\frac{1}{2}}[(1+\beta)n+(2\delta-1)\beta+1]^{\frac{1}{2}}}{\sqrt{2\beta(1-\alpha)}\sqrt{2\beta(1-\delta}\,[(1+\beta)n+(2\rho-1)\beta+1]}. \tag{5.23}$$

It follows from (5.23) that

$$\rho \leq 1-\frac{2\beta(1+n)(1+\beta)(1-\alpha)(1-\delta)}{n^k[(1+\beta)n+(2\alpha-1)\beta+1][(1+\beta)n+(2\delta-1)\beta+1]+4\beta^2(1-\alpha)(1-\delta)}.$$

Now defining the function $M(n)$ by

$$M(n) = 1-\frac{2\beta(1+n)(1+\beta)(1-\alpha)(1-\delta)}{n^k[(1+\beta)n+(2\alpha-1)\beta+1][(1+\beta)n+(2\delta-1)\beta+1]+4\beta^2(1-\alpha)(1-\delta)},$$

we see that $M(n)$ is an increasing function of $n(n \geq 1)$. Therefore, we conclude that

$$\rho \leq M(1) = 1-\frac{\beta(1+\beta)(1-\alpha)(1-\delta)}{(1+\alpha\beta)(1+\delta\beta)+\beta^2(1-\alpha)(1-\delta)}$$

and hence the proof of Theorem 11 is completed.

Theorem 12. *Let the function* $f_1(z)$ *defined by (3.4) be in the class* $\sum C^*(k,\alpha,\beta)$. *Suppose also that the function* $f_2(z)$ *defined by (3.4) be in the class* $\sum C^*(k,\delta,\beta)$. *Then* $(f_1 * f_2)(z) \in \sum C^*(k,\rho,\beta)$, *where* ρ *is given by (5.14). Sharpness holds for functions given by (5.15) and (5.16), respectively.*

Theorem 13. *Let the functions* $(f_j)(z)(j=1,2)$ *defined by (3.4) be in the class* $\sum S^*(k,\alpha,\beta)$. *Then the function*

$$h(z)=\frac{1}{z}+\sum_{n=1}^{\infty}(a^2{}_{n,1}+a^2{}_{n,2}) \qquad (5.24)$$

belongs to the class $\sum S^*(k,\zeta,\beta)$ *, where*

$$\zeta=1-\frac{2\beta(1+\beta)(1-\alpha)^2}{(1+\alpha\beta)^2+2\beta^2(1-\alpha)^2} \qquad (5.25)$$

The result is sharp for functions $(f_j)(z)(j=1,2)$ *defined by (5.3).*

Proof. By using Lemma le1, we obtain

$$\sum_{n=1}^{\infty}\left\{\frac{n^k[(1+\beta)n+(2\alpha-1)\beta+1]}{2\beta(1-\alpha)}\right\}^2 a^2{}_{n,1}$$

$$\leq\sum_{n=1}^{\infty}\left\{\frac{n^k[(1+\beta)n+(2\alpha-1)\beta+1]}{2\beta(1-\alpha)}a_{n,1}\right\}^2\leq 1 \qquad (5.26)$$

and

$$\sum_{n=1}^{\infty}\left\{\frac{n^k[(1+\beta)n+(2\alpha-1)\beta+1]}{2\beta(1-\alpha)}\right\}^2 a^2{}_{n,2}$$

$$\leq\sum_{n=1}^{\infty}\left\{\frac{n^k[(1+\beta)n+(2\alpha-1)\beta+1]}{2\beta(1-\alpha)}a_{n,2}\right\}^2\leq 1 \qquad (2.27)$$

It follows from (5.26) and (5.27) that

$$\sum_{n=1}^{\infty} \frac{1}{2} \left\{ \frac{n^k[(1+\beta)n + (2\alpha - 1)\beta + 1]}{2\beta(1-\alpha)} \right\}^2 (a^2{}_{n,1} + a^2{}_{n,2}) \leq 1.$$

Therefore, we need to find the largest ζ such that

$$\frac{n^k[(1+\beta)n + (2\zeta - 1)\beta + 1]}{2\beta(1-\zeta)} \leq \frac{1}{2} \left\{ \frac{n^k[(1+\beta)n + (2\alpha - 1)\beta + 1]}{2\beta(1-\alpha)} \right\}^2,$$

that is

$$\zeta = 1 - \frac{4\beta(1+n)(1+\beta)(1-\alpha)^2}{n^k[(1+\beta)n + (2\alpha - 1)\beta + 1]^2 + 8\beta^2(1-\alpha)^2}.$$

Now defining the function $H(n)$ by

$$H(n) = 1 - \frac{4\beta(1+n)(1+\beta)(1-\alpha)^2}{n^k[(1+\beta)n + (2\alpha - 1)\beta + 1]^2 + 8\beta^2(1-\alpha)^2},$$

we see that $H(n)$ is anincreasing function of $n(n \geq 1)$. Therefore, we conclude that

$$\zeta \leq H(1) = 1 - \frac{2\beta(1+\beta)(1-\alpha)^2}{(1+\alpha\beta)^2 + 2\beta^2(1-\alpha)^2}.$$

and hence the proof of Theorem 13 is completed.

Theorem 14. *Let the functions* $(f_j)(z)(j = 1,2)$ *defined by (3.4) be in the class* $\sum C^*(k, \zeta, \beta)$ *Then the function* $h(z)$ *given by (5.24) belongs to the class* $\sum C^*(k, \zeta, \beta)$ *, where* ζ *is given by (5.25). The result is sharp for functions* $(f_j)(z)(j = 1,2)$ *defined by (5.3).*

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