The Effect of Air Resistance on projectiles Motion equation and study General Solution with air resistance quadratic in the speed

[*]Marzouga F.Rajab [**]Mizouna.G Asbig [**] Manal.K Zaid [**]Hager .A.Mohamed

Abstract: This research work study the a projectiles motion without air resistance and projectiles motion with air resistance by using vector-valued function. We consider two-dimensional motion of a projectile experiencing a constant gravitational force in the projectile's speed and look at launching projectiles, as well as how varying initial velocity and height affect the launch angle. Finally, we add air resistance to the projectile problem and compare two different models: air resistance proportional to the projectile's velocity and air resistance proportional to velocity squared

We find that :The equations of motions are coupled nonlinear equations. Their solutions have general properties which are easily visualized, although much different from those obtained .when an air resistance is neglected

Introduction:

Most introductory physics courses spend a considerable amount of time studying the motion of projectiles but almost always ignore the air resistance that inevitably impacts the motion of these objects. In many problems this is an excellent approximation; in others, air resistance is obviously very important and we need to know how to account for it. In this paper we will investigate General Solution of motion equation with air resistance quadratic in the speed, but at first, we try to make compare between the equation of motion with air -resistance and .without it and how air resistance effects in the motion of falling objects and projectiles

Equations of motion: no air resistance:

We assume that gravity is the only force acting on the projectile after it is launched. So, the motion occurs in a vertical plane, which can be represented by the xy-coordinate system with .the origin as a point on Earth's surface, as shown in Figure 1

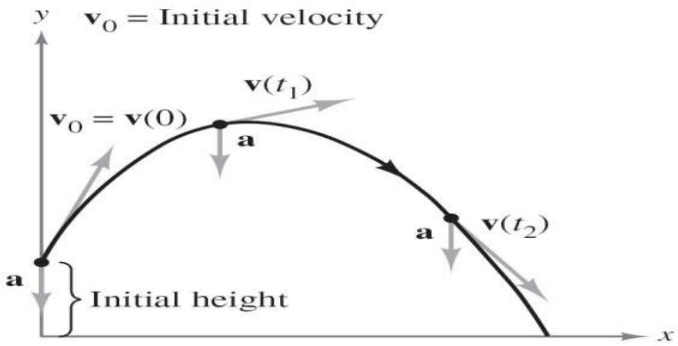

Figure 1

For a projectile of mass m, the force due to gravity is [1]

$\mathbf{F} = -mg\boldsymbol{j}$ Force due to gravity …………..)1(

---

[*] Department of physics, Faculty of Science, Omar El-Mukhtar University, Elgubba. Libya marzouga12@yahoo.com

[**] Department of physics, Faculty of Science, Omar El-Mukhtar University, Elgubba. Libya

[**] Department of physics, Faculty of Science, Omar El-Mukhtar University, Elgubba. Libya

[**] Department of physics, Faculty of Science, Omar El-Mukhtar University, Elgubba. Libya

where the acceleration due to gravity is g = 32 feet per second per second, or 9.81 meters .per second per second

By Newton's Second Law of Motion, this same force produces an acceleration

$\mathbf{a} = \mathbf{a}(t)$, and satisfies the equation [2]

$\mathbf{F} = m\mathbf{a}$……………….)2(

Consequently, the acceleration of the projectile is given by

$$m\mathbf{a} = -mg\boldsymbol{j}$$

which implies that

$\mathbf{a} = -g\boldsymbol{j}$. Acceleration of projectile

We start by finding a position vector as a function of time (t). Beginning with the acceleration vector

$\mathbf{a} = -g\boldsymbol{j}$ and integrating twice

$$\mathbf{v}_t = \int \mathbf{a}(t)dt = \int -g\boldsymbol{j}dt = -gt\boldsymbol{j} + \boldsymbol{C}_1$$

$$\mathbf{r}_t = \int \mathbf{v}(t)\,dt = \int (-gt\boldsymbol{j} + \boldsymbol{C}_1)\,dt = -1/2\,gt^2\,\boldsymbol{j} + \boldsymbol{C}_1 t + \boldsymbol{C}_2$$

Solving for the constant vectors $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$, we use the fact that

$$\mathbf{v}(0) = \mathbf{v}_0 \quad \text{and} \quad \mathbf{r}(0) = \mathbf{r}_0$$

$$\boldsymbol{C}1 = \mathbf{v}0 \quad \text{and} \quad \boldsymbol{C}2 = \mathbf{r}0$$

Therefore, the position vector is

position vector ……………..)3($\mathbf{r}_t = -\frac{1}{2}gt^2j + tv_0 + r_0$

In many projectile problems, the constant vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ are not given explicitly. Often, we are given the initial height h, the initial speed v_0 and the angle θ at which the projectile is launched, as shown in Figure 2

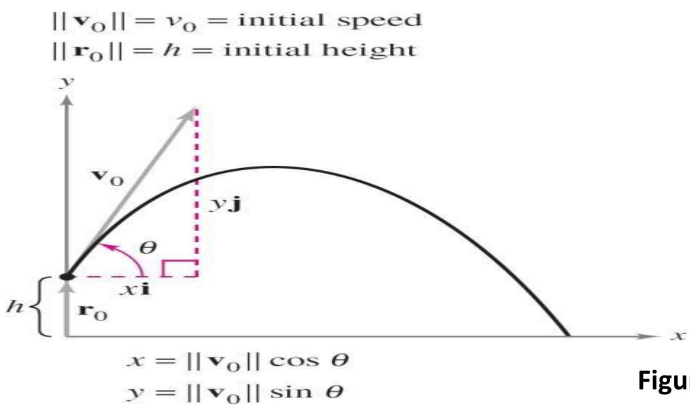

Figure 2

From the given height, we can deduce that

$$\mathbf{r}_0 = h\boldsymbol{j}$$

Because the speed gives the magnitude of the initial velocity, it follows that

$= v_0$

and we can write

$\|v_0\|$

$$v_0 = x\boldsymbol{i} + y\mathrm{j}$$

$$= (\|v_0\|\cos\theta)\boldsymbol{i} + (\|v_0\|\sin\theta)\boldsymbol{j}$$

$$= v_0\cos\theta\boldsymbol{i} + v_0\sin\theta\boldsymbol{j}$$

So, the position vector can be written in the form

$$\mathbf{r}_t = -\frac{1}{2}gt^2j + tv_0 + r_0 \qquad \text{position vector} \ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

$$= -\frac{1}{2}gt^2j + tv_0\cos\theta i + tv_0\sin\theta j + hj$$

$$= v_0(\cos\theta)\,t\boldsymbol{i} + [\,h(v_0\sin\theta)t - \frac{1}{2}gt^2]\,j$$

Neglecting air resistance, the path of a projectile launched from an initial height h with initial speed v_0 and angle of elevation θ is described by the vector function[1,3]

$$r(t) = v_0(\cos\theta)\,t\boldsymbol{i} + [\,h(v_0\sin\theta)t - \frac{1}{2}gt^2]\,j \quad \ldots\ldots\ldots\ldots\ldots\ldots..(5)$$

where g is the acceleration due to gravity.

The Normal and Binomial Vectors: At a given point on a smooth space curve $\mathbf{r}(t$

The Normal and Binomial Vectors: At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$ .We single out one by observing that, because

$\mathbf{T}(t) = 1$ for all t,

we have $\mathbf{T}(t)\cdot\mathbf{T}'(t) = 0$

so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$.

Note that $\mathbf{T}'(t)$ is itself not a unit vector.

But if $\mathbf{r}'$ is also smooth, we can define the principal unit normal vector $\mathbf{N}\ t$ (or simply unit normal) as

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots.(6)$

is called the binomial vector.

It is perpendicular to both **T** and **N** and is also a unit vector as shown in Figure 3

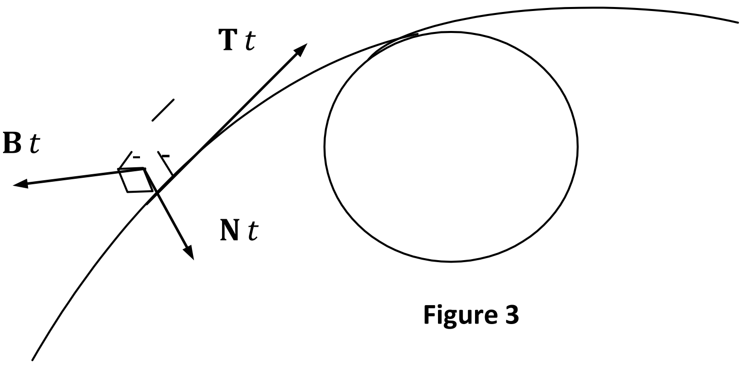

Figure 3

We can think of the normal vector as indicating the direction in which the curve is turning at .each point

Equations of motion: linear air resistance

In order to make the projectile problem more realistic, we consider the effects of air resistance. Air resistance is a force, called the drag force, that acts in the direction opposite an object's motion. Air resistance takes the form of a Taylor series where the terms are powers of the projectile's velocity. Overall, two of the terms tend to be much larger than all .the other terms: the v and the v^2 terms

For this reason, we consider two main types of drag force: linear and quadratic. These forces take the form [4]

$$F_{lin} = bv \qquad F_{quad} = cv^2$$

where b and c are constants with $b = \beta D$ and $c = \gamma D^2$, where D is the diameter of the object and β and γ are constants that depend on the nature of the medium.An object subject to air resistance is modeled by a linear combination of both linear and quadratic drag forces, but in many cases, one term is much larger than the other, indicating the situation can be modeled as either linear or quadratic. In order to understand which drag force will be most realistic for the problem at hand, we estimate the ratio F_{quad} / F_{lin}, which tells us if one model of air .resistance can be neglected

When solving the projectile problem, we first consider linear air resistance in order to show how air resistance is included in projectile motion. We choose this model because mathematically it is simpler and provides insight into modeling air resistance. Quadratic air resistance is a more realistic model for our situation, but we will see that the equations of .motion in this case are not solvable in a closed form using differential equations techniques

We therefore turn to computer approximations to model this motion, but first let us further .explore the linear model

when we explored the case without air resistance, we solved second order differential equations for the x- and y-motion of the projectile as functions of time (t). The method here is the same, except we must consider a force other than gravity acting on the projectile. Drag force acts in the direction opposing motion, so both the x- and y equations will have a component of air resistance. Our second order differential equations will be the x- and ycomponents of acceleration, which we find from Newton's 2nd law. Bolded values indicate .a vector, meaning the variable has both a magnitude and direction
We define the drag force to be F_D and the gravitational force is F_g. We have[1]

(7)……………. ma = F = $F_g + F_D$ = mgy - b(xˆ + yˆ)

and letting k = b/m, we can separate the above equation into x- and y-equations. We have

$$\bar{\bar{x}}(t) = -k\bar{x}(t)$$

$$\bar{\bar{y}}(t) = -g - k\bar{y}(t)$$

$$x(0) = 0\,; \qquad y(0) = h;$$

$$\bar{x}(0) = v\cos\theta; \qquad \bar{y}(0) = v\sin\theta;$$

is the initial velocity of the projectile. Using separation of variables to solve the x v where equation, we obtain

$$\bar{\bar{x}}(t) = -k\bar{x}(t)$$

$$\bar{x}(t) = Ce^{-kt} = v\cos\theta\, e^{-kt}$$

.)8(................ $x(t) = -\frac{v\cos\theta}{k}\, e^{-kt} + C = \frac{v\cos\theta}{k}\,(1 - e^{-kt})$

Similarly, for the y-equation, we have

$$\bar{\bar{y}}(t) = -g - k\bar{y}(t)$$

$$\frac{d\bar{y}}{-g - k\bar{y}} = dt$$

$$\frac{1}{k}\ln(g + k\bar{y}) = -t + C$$

$$(g + k\bar{y}) = Ce^{-kt}$$

، We solve for C using the initial condition $\bar{y}(0) = v\sin\theta$

g + k(v sin θ) = C

g - k $\bar{y}$=)]-g- kv sinθ(e^{-kt}[

$$\bar{y} = \frac{1}{k}\,[-g + (g + kv\sin\theta)e^{-kt}]$$

Using the initial condition y(0) = h, we integrate once more with respect to t to find the motion equation for y:

g g □kt)c
y □)□ t□)□ vsin□ e k(k

$$y = h + \frac{1}{k}(v\sin\theta - gt) + \frac{1}{k^2}\,[g - e^{-kt}(g + kv\sin\theta)] \;\ldots\ldots\ldots\ldots\ldots\ldots.(9)$$

Now that we have the x- and y-equations of motion, we want to eliminate the variable t, by solving the x-equation for t and substituting this t-equation into the y-equation. We have

$$x(t) = \frac{v\cos\theta}{k}\left(1 - e^{-kt}\right)$$

$$-\frac{1}{k}\ln\left(\frac{-xk}{v\cos\theta} + 1\right) \text{ t=}$$

which gives us

$$y = h + \frac{1}{k}\left[v\sin\theta + g\ln\left(\frac{-xk}{v\cos\theta} + 1\right)\right] + \frac{1}{k^2}\left[-(g + kv\sin\theta)\right]\left(\frac{-xk}{v\cos\theta} + 1\right) \ldots (10)$$

We now have a function for the projectile's path in terms of x.

EQUATIONS OF MOTION AND SOME GENERAL PROPERTIES OF THEIR :SOLUTIONS

Physics students are certainly familiar with the solution of projectile problems when air resistance is neglected. Since, in practice, air resistance is usually not negligible, it would be of some interest to develop solutions of the equations of motion including an appropriate drag force. If one assumes a drag force which is linear in the speed, then it is straightforward to solve the equations of motion. Such a drag force actually occurs for low .and is dependent upon the viscosity of the fluid ،($\mathcal{R}>1$) ،Reynolds numbers

To determine whether or not it applies, one must calculate the Reynolds number. For a the viscous drag force and ،η sphere of radius r moving in a fluid of density ρ and viscosity the Reynolds number are[5, 6]

$$F = 6\pi\eta r v \quad for\ \mathcal{R} > 1,$$

$$\mathcal{R} = 2r\rho V\eta^{-1} \text{ (11)……..}$$

The terminal speed U is then given by

$$6\pi\eta r U = mg = \frac{4\pi r^3\rho_s\, g}{3}$$

is the sphere's density. In order that Eq. (11) apply to the whole of a trajectory, ،ρ_s where for speeds up to the terminal speed. This can be considered a minimum $\mathcal{R} \leq 1$ we require requirement since projectiles are often started off with speeds greater than this. Assuming a cm^2/sec for 0.15 / $\eta\rho =$ sphere which has the density of water, 1 g/cm^3, and using the value air, one finds that Eq. (11) applies when [6]

$$r \geq 4\times10^{-3}\text{cm} \ldots\ldots\ldots)12($$

Spheres this small would seem to be of little interest. In fact, these small radii are not that far from the mean free path λ of air molecules, which is typically of the order of 10^{-5} cm. A second criterion for the validity of Eq. (11) is[7]

$r >> \lambda$ …………)13(

We conclude from Eqs. (12) and (13) that Eq. (11) almost never applies exclusively. It is known that for spheres with radii and speeds of practical interest a reasonable approximation to the drag force is:[8]

$$F = \frac{C_D\, \rho A v^2}{2} \quad \text{……………. (14)}$$

The drag coefficient is

$$C_D \approx \frac{1}{2}$$

And

$$\pi r^2 = A$$

The equations of motion including the drag force of Eq.(14) are more difficult to solve, however. An exact and explicit solution for vertical free fall, or one-dimensional motion, can be obtained rather easily[9]. For two-dimensional motion, however, no such solution appears to exist. This offers an opportunity to develop approximate solutions to a nonlinear problem of some pedagogical interest. Thus, in this paper, we derive some simple approximate solutions for both long and short times. In addition, an exact but implicit solution is developed which enables accurate numerical solutions to be calculated for comparison. Finally, we present a description of the quadratic drag force which focuses attention on the kind of turbulence that is known to be associated with this force law.[10] From that, the equations of motion are

$$m\frac{dv_x}{dt} = -bv_x\,(v_x^2 + v_y^2)^{1/2}$$

$$m\frac{dv_y}{dt} = -mg - bv_y\,(v_x^2 + v_y^2)^{1/2}$$

We have chosen the positive y direction to be vertically upward and the constant b to represent the coefficient in Eq. (14). The natural scale of velocities is set by the terminal speed V, which corresponds to the solution $v_x = 0 \; and \; v_y = -V$ ،Thus،

$$V = \left(\frac{mg}{b}\right)^{1/2}$$

This is the only combination of the available parameters which has velocity units. Given this speed and the acceleration of gravity, the natural time unit is

$$\frac{V}{g} = T$$

This is a measure of the time required to reach terminal speed. With these two units all subsequent equations may be expressed in dimensionless form. To this end we define scaled time, velocity, and displacement variables as follows

$$\tau = \frac{t}{T}$$

$$u_x = \frac{v_x}{V} \text{ ، } u_y = \frac{v_y}{V}$$

$$X = \frac{x}{VT}, Y = \frac{y}{VT}$$

In terms of these new variables, the equations of motion are

$$\dot{u}_x = -u_x\,(u_x^2+u_y^2)^{1/2}$$

$$\dot{u}_y = -1 - u_x\,(u_x^2+u_y^2)^{1/2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

The new $.\tau$ with respect to(τ)Here we have used dots to denote derivatives of u_x( ) and u_y displacements are

$$X = X_0 + \int_0^\tau u_x d\tau$$

$$Y = Y_0 + \int_0^\tau u_y d\tau$$

or speeds ، τ It will be seen that for times

$$u =)u_x^2 + u_y^2)^{1/2} \quad (16)\ldots\ldots\ldots\ldots\ldots\ldots،$$

or displacements X or Y small compared to unity the effects of air resistance will be small.

Discussion

:We describe here some general properties of the solutions of Eq.)15(

1. The most often noted feature of Eq. (15) is the existence of the terminal speed V. Hail .stones are a good example to use. Their speed with no drag would make them lethal
2. No matter what the velocity is initially, the final velocity is always the same, namely u_x=0, .u_y= - 1
3. Every path in the x,y plane corresponding to different initial velocities has a different shape. In the absence of drag every path has the same parabolic shape. Thus, the higher the initial speed of a projectile for a given upward projection angle the more blunted the forward .end of its path becomes, i.e., the less symmetrical it is about its peak
4. The maximum horizontal range for a given initial speed occurs at angles less than

and the greater the initial speed, the lower the projection angle that is required. For ،°45 example, we find that the maximum horizontal range occurs at 36° for an initial speed u_0 = .2

5. Since ug approaches zero from its initial value u_{x0} , there is a maximum horizontal displacement which will be of the order of u_{x0} in terms of scaled variables (i.e., $v_{x0} T = u_{x0} V T$). Similarly, if the projectile is fired upward, it will reach a maximum y dis- placement of the order of u_{y0}. where u_{y0} is the initial value of u_y

المستخلص: يدرس هذا العمل البحثي حركة المقذوفات بدون مقاومة للهواء وحركة المقذوفات مع مقاومة للهواء باستخدام دالة ذات قيمة متجهة. حيث أننا نأخذ في الاعتبار الحركة ثنائية الأبعاد لقذيفة تتعرض لقوة جاذبية ثابتة في سرعة القذيفة وننظر في إطلاق هذه المقذوفات ، وكذلك كيف تؤثر السرعة الأولية والارتفاع المتغيران على زاوية الإطلاق. أخيرًا ، نضيف مقاومة الهواء لحل مشكلة القذيفة ونقارن بين نموذجين مختلفين: مقاومة الهواء المتناسبة مع سرعة المقذوف ومقاومة الهواء المتناسبة مع مربع السرعة

نجد أن: معادلات الحركة مقترنة معادلات غير خطية. تتميز حلولهم بخصائص عامة يمكن تصورها بسهولة ، على الرغم من اختلافها كثيرًا عن تلك التي يتم الحصول عليها عند إهمال مقاومة الهواء

:References:

1. Nina Henelsmith, Projectile Motion: Finding the Optimal Launch Angle (Whitman .(College May 12, 2016
2. Mohd Faraz ,Study of projectile motion with air resistance, September 2018

CITATIONS

3. Richmond Opoku-Sarkodie and Edward Acheampong,Vector-Valued Function Application to Projectile Motion ,May 2015

4. Chokri Hadj Belgacem, Analysis of projectile motion with quadratic air resistance from a nonzero height using the Lambert W function, Journal of Taibah University for Science . (2016)

5. G. K. Batchelor, An Introduction to Fluid Dynamics (Cambridge U.P., London, 1967), .pp. 233—234

6. G. W. Parker .Projectile motion with air resistance quadratic in the speed (American .Journal of Physics · July 1977) DOI: 10.1119/1.10812

7. R. D. Present, Kinetic Theory of Gases (McGraw-Hill, New York, 1958), p. 162'

8. .Reference 5, pp. 339—341

9. J. Lindemutt, Am. J. Phys. 39, 757)1971(

10. A'.A. P. French, Newtonian Mechanics (Norton, New York, 1971), pp. 213—214