A survey of half-groups

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Abstract: In this paper, we introduce the idea of half-groups. This is an absolutely new idea and deserves significant consideration. We have imposed a group structure on a half groupoid wherein we have an identity element and each element having a unique inverse under the specified operation, then, at that point, we addressed half groups, subhalf-groups, and normal subhalf-groups. All of the fundamentals discussed in this research were backed by instances and lemmas to demonstrate their validity.

Key words: half -groups , subhalf-groups, normal subhalf- groups.

Introduction:

Furstenberg [1]and Whittake[2] among others,have exhibited postulate systems for groups in terms of the operation$x - y = x + (-y)$. Furstenberg also investigated a system obtained byremoving one of his postulates which defines what he called a half-group.

Astructure theorem for half-groups was given in [1], In the present paper, we introduced another new structure theorem for half-groups.[4].

Definition .1.1.[4]Let$(H,*)$be a half groupoid (a partially closed set with respect to *) such that

(1) There exists an element $e \in H$,such $a * e = e * a = a$,$\forall a \in H$, e is calledidentity element of H

(2) For every $a \in H, \exists\ b \in H$such that $a * b = b * a = e$,b is called the inverseof a.

Then$(H,*)$ is called a half-group.

Lemma.1.2: Let (H , *) be a half-group , then :

(1) The identity element of H is unique.

(2) Every $a \in H$has a unique inverse in H

(3) For all $a \in H, (a^{-1})^{-1} = a$.

(4) associativity does not hold in H, as there is at least one product that is not defined in H.

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Proof:

(1) Let H be a half-group with two identities $e,f \in H$

$\forall\, a \in H\,, e*a = a*e = a$

and $a*f = f*a = a$

$\Rightarrow a = a*e = a*f \Rightarrow e = f$. The identity is unique.

(2) Suppose that$x \in H,\ a*x = e\ \ and\ \ a*y = e$,

thenobviously

$a*x = a*y\ for\ a,x,y\ \in H, there\ is\ an\ element\ e \in H\ s,t$

$a*x = a*y = e\ \ by\ cancelation\ law\ \Rightarrow x = y.$

(3) Let $a^{-1}\ \ is\ the\ inverse\ of\ (a^{-1})^{-1}$, $(and\ we\ know\ \ e = a^{-1}*a\,)$

$\Rightarrow\ a^{-1}*(a^{-1})^{-1} = e = a^{-1}*a$

$\Rightarrow a^{-1}*(a^{-1})^{-1} = a^{-1}*a$

$by\ cancelation\ law\ \Rightarrow \qquad (a^{-1})^{-1} = a.$

Note:In next tables in the following examples the blank entries show that the corresponding products are not defined.

Example (1):[4] Let $S = \{1, -i\,, i\}\,, the\ \ S$ is a half- group w.r.t

multiplication,

*	1	-i	i
1	1	-i	i
-i	-i		1
i	i	1	

$(-i(i) = -(-1) = 1 \in S, -i*-i = -1\ \notin S\,, and\ \ i*i = -1\ \notin S)$

Definition1.3[4] .Let $(H,*)$ be a half-group and g a subset of H, if g

itselfisa half-groupw.r.t (*) the g is called a sudhalf-group of H.

Example(2).[4] Let $S = \{e,a,b,c,d\}\,, then\ (\ S,*) is\ a\ half - group$,

defined by the following table, (d*c is not defined)

*	e	a	B	c	d
e	e	a	B	c	d
a	a	c	e	b	a
b	b	e	c	a	d
c	c	d	a	e	b
d	d	a	c		e

then $H=\{e,a,b\}$ is a subhalf-group of S.since

*	e	a	b
e	e	a	b
a	a		e
b	b	e	

$(a*a=c \notin H, \quad and \ b*b=c \notin H)$

Definition1.4: A subhalf-group H of a half-group G is called Normal-subhalf-group of Gif:

$gh=hg \ for \ every \ g \in G$.

since$g^{-1} \in G \ when \ ever \ g \in G$.

may replaced by (1) $g^{-1}Hg=H \ for \ every \ g \in G$.

Now (1) requires:

(i) $for \ any \ g \in G \ and \ any \ h \in H, then \ g^{-1} \circ h \circ g \in H$.

(ii) $for \ any \ g \in G \ each \ h \in H, there \ exists \ some \ k \in H \ s.t$

$g^{-1} \circ k \circ g=h \ , or \ \ k \circ g=g \circ h,$

we shall show that (i) implies (ii)

Consider any $h \in H \ , \ \ by \ (i)(g^{-1})^{-1} \circ h \circ g^{-1}=g \circ h \circ g^{-1}=k \in H.$

$since \ g^{-1} \in G \ ,then \ g^{-1} \circ k \circ g=h \ ,as \ requierd.$

Examples(3): Every subhalf-group H of an abelian half- group G is normal subhalf-group of G: since

$g \circ h=h \circ g \ for \ any \ g \in G \ and \ every \ h \in H$.

lemma1.5 : H is normal subhalf-group of half-group G, if and only if $gHg^{-1}=H \ for \ every \ g \in G.$

Proof : if $gHg^{-1}=H, for \ every \ g \in G$.

certainly $gHg^{-1} \subset H,$ so H is normal subhalf-group in G.

suppose H is normal subhalf-group in G,

Thus if $g \in G, \ gHg^{-1} \subset H, \ and$

$g^{-1}Hg=g^{-1}H(g^{-1})^{-1} \subset H.$

now, since $g^{-1}Hg \subset H \ ,H=g(g^{-1}Hg)g^{-1} \subset gHg^{-1} \subset H,$

whence$H=gHg^{-1}$.

تقدمة عن نصف الزمر

المستخلص: في هذه الورقة البحثية نستعرض مفهوم نصف الزمرة ، و هو مفهوم حديث في تعريفه و تطوره حيث قدمنا في البداية تركيبة الزمرة الاولية ذات العنصر المحايد و المعكوس الوحيد لكل عنصر من عناصر الزمرة تحت نفس العملية ، تحديداً قدمنا التعريف الحديث لنصف الزمرة، و نصف الزمرة الفرعية ، و نصف الزمرة الفرعية العادية، ومن خلال هذه الورقة قدمنا التعريفات المهمة السابقة و أثبتنا بعض النظريات المتعلقة بموضوع االبحث.

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