On Some Properties of Solutions of Second Order Rational Recursive Sequences

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Abstract: Equations that enable us to iteratively calculate a function's value from a given set of values are commonly the foundation of mathematical computations. Such as equations are known as recurrence equations or difference equations. In this paper, we study some known properties of this kind of equations. Some new results are provided which are consistent with several results from earlier researches that deal with second order rational recursive difference equations.
Keywords. Rational Recursive Sequences; Difference Equations; Second Order, Global Attractivity; Asymptotic.

المستخلص: المعادلات التي تمكننا من حساب قيمة دالة بشكل متكرر من مجموعة معينة من القيم هي عادة أساس الحسابات الرياضية. تُعرف مثل هذه المعادلات بمعادلات التكرار أو معادلات االفروق. وفي هذا البحث قمنا بدراسة بعض الخصائص المعروفة لهذا النوع من المعادلات. تم تقديم بعض النتائج الجديدة والتي تتوافق مع العديد من نتائج الأبحاث السابقة التي تناولت معادلات الفروق التكرارية من الرتبة الثانية.
الكلمات المفتاحية: المتتابعات التكرارية القياسية. معادلات الفروق. رتبة ثانية. الجاذبية العالمية. التقارب.

Introduction

Over the past few decades, the study of the global attractivity, asymptotic stability and oscillatory features and some other properties of solutions of difference equations have drawn a lot of interest (see Amleh et al. [2], Graef and Qian [8], Kocic [12], Stevic [16] and Yang et al. [18]). The reason for this is that, the difference equations are suitable models for describing situations in which the variable is assumed to only occasionally occur in the study of biological models in the formulation and analysis of discrete-time systems and the numerical integration by finite difference schemes. Readers are encouraged to read the articles for further explanation of Devault et al. [4], Hamza [9], Hamza and Sayed [10] and Kulenvic et al. [13] and Philos et al. [14]. For the theory of the difference equations in general, see the monographs of Agarwal [1], Elaydi [5], Kelly et al. [11] and Ronald [15].

However, our result includes certain aspects of the qualitative theory of difference equation: the asymptotic behavior and global attractiveness of the solutions to the difference equations of the forms

$$x_{n+1} = \frac{\alpha+\beta x_n^q+\gamma x_{n-1}^q}{A+Bx_n^p+Cx_{n-1}^p}, n = 0,1, \dots\dots \qquad (E_1)$$

where the parameters $A, B, C, \alpha, \beta, \gamma$ and the initial conditions x_{-1} and x_0 are real numbers such that $A + Bx_n^p + Cx_{n-1}^p > 0$ for all $n \geq 0, p\ and\ q\ \in N\ and$

$$x_{n+1} = \frac{a\,x_n+bx_{n-1}}{c+dx_n^p x_{n-1}^p} \qquad (E_2)$$

where $a \geq 0, b, c, d > 0\ and\ p\ \in N.$

As is well known, studying the behavior of difference equation solutions is an important issue. Specially; in the study of concrete dynamical processes. Such as this behavior is influenced in many instances by the initial conditions associated to the given difference equations. (see Berinde [3] and El-Owaidy et al. [7])

To the best of our knowledge, the aim of the current paper is to study the asymptotic stability and the global attractivity properties of solutions for a class of difference equations with hope

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to generate an interesting new direction of further work in future following the valuable ideas of the excellent papers of El-Morshedy [6] and Yan et al. [17].

The results of this work, at least in our opinion, are extremely significant on their own rights and provide a foundation for the future development of a basic theory of the global behavior of second-order non-linear difference equation solutions. Equations in mathematical models of different biological systems and other applications might be analyzed with great benefit by using the methods and findings of such this work.

Let I be some interval of real numbers and let

$$f : I \times I \to I$$

be continuously differentiable function. Then for every set of initial conditions $x_0, x_{-1} \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}), n = 0,1,\ldots.. \qquad (E_3)$$

has a unique solution $\{x_n\}_{n=-1}^{\infty}$.

Definition 1. A point $\bar{x} \in I$ is called an equilibrium point of the equation (E_3) if $\bar{x} = f(\bar{x}, \bar{x})$, which means that

$$x_n = \bar{x} \text{ for all } n \geq 0$$

is a solution of (E_3) or equivalently, $\bar{x}$ is a fixed point of the function f .

Definition 2. Let $\bar{x}$ be an equilibrium point of (E_3), then we have the following:

(i) $\bar{x}$ is stable if for every $\in > 0$, there exists $\delta > 0$ such that for any initial conditions

$$(x_{-1}, x_0) \in I \times I$$

with

$$|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \delta$$

we have

$$|x_n - \bar{x}| < \varepsilon \text{ for all } n \geq -1.$$

(ii) $\bar{x}$ is a local attractor if there exists $\gamma > 0$ such that $x_n \to \bar{x}$ holds for any initial conditions

$$(x_{-1}, x_0) \in I \times I$$

with

$$|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma.$$

(iii) $\bar{x}$ is locally asymptotically stable if it is stable and is a local attractor.

(iv) $\bar{x}$ is a global attractor if $x_n \to \bar{x}$ holds for any initial conditions $(x_{-1}, x_0) \in I \times I$.

(v) $\bar{x}$ is globally asymptotically stable if it is stable and is a global attractor.

(vi) $(x_{-1}, x_0) \in I \times I$ such that for each $\gamma > 0$ re exists is arepeller if the $\bar{x}$

with

$$|x_{-1} - \bar{x}| + |x_0 - \bar{x}| < \gamma,$$

there exists $N \geq 1$ such that

$$|x_N - \bar{x}| \geq \gamma.$$

(vii) $\bar{x}$ is a saddle point if it is neither a local attractor nor arepeller.
(viii) $\bar{x}$ is unstable if $\bar{x}$ is not stable.

Definition 3. Assume that

$$p_1 = -\frac{\partial f(\bar{x},\bar{x})}{\partial u} \quad \text{and} \quad q_1 = -\frac{\partial f(\bar{x},\bar{x})}{\partial v},$$

Denote to the partial derivatives of $f(u,v)$ evaluated at an equilibrium point $\bar{x}$ of (E_3). Then the equation

$$y_{n+1} + p_1 y_n + q_1 y_{n-1} = 0, n = 0,1,2, \ldots\ldots \qquad (E_4)$$

is called the linearized equation associated with (E_3) about the equilibrium point $\bar{x}$. Therefore, its characteristic equation is

$$\lambda^2 + p_1\lambda + q_1 = 0 \qquad (E_5)$$

In the following, we consider the asymptotic stability for the recursive sequence.

(a) The equilibrium point $\bar{x}$ of (E_3) is locally asymptotically stable if the all solutions of the (E_5) lie inside the open unit disk $|\lambda| < 1$.

(b) The equilibrium point $\bar{x}$ of (E_3) is unstable if at least one of the solution of (E_5) has an absolute value larger than one.

(c) The $|p_1| < 1 + q_1$ and $q_1 < 1$ are necessary and sufficient conditions for all the solutions of (E_5) to lie inside the open unit disk $|\lambda| < 1$. Hence, in this case the locally asymptotically stable equilibrium $\bar{x}$ is also referred as a sink.

(d) The $|q_1| > 1$and $|p_1| < |1 + q_1|$ are are necessary and sufficient conditions for all the solutions of (E_5) to have absolute value greater than one. Therefore, in this case $\bar{x}$ is referred as a repeller.

(e) The $p_1^2 > 4q_1$ and $|p_1| > |1 + q_1|$ are necessary and sufficient conditions for one solution of (E_5) to have absolute value greater than one and for the other solution to have absolute value less than one . Therefore, in this case the unstable equilibrium point $\bar{x}$ is referred a saddle point.

(f) The $|p_1| = |1 + q_1|$ or $q_1 = 1$ *and* $|p_1| \le 2$ are necessary and sufficient condition for the solution of (E_5) to have absolute value equal to one. Hence, in this case the equilibrium point $\bar{x}$ is referred as a non-hyperbolic point.

Main Results

In this section, we are interested in studying the global stability of the recursive sequence

$$x_{n+1} = \alpha + \frac{x_n}{x_{n-1}}, \qquad n = 0,1, \ldots \qquad (1.1)$$

where $\alpha \in \left[\left(-2, -\frac{3}{4}\right) - \{-1\} \cup (0.\infty)\right]$ and the initial conditions x_{-1} *and* x_0 are negative numbers.

The Linearized equation associated with (1.1) about the equilibrium point $\bar{x} = \alpha + 1$ is

$$y_{n+1} - \frac{1}{\alpha+1} y_n + \frac{1}{\alpha+1} y_{n-1} = 0, \quad n = 0,1, \ldots \qquad (1.2)$$

The characteristic equation of (1.2) is

$$\lambda^2 - \lambda\theta + \theta = 0 \qquad (1.3)$$

Where

$$\theta = \frac{1}{\alpha + 1}$$

The solution of (1.3) are

$$\lambda_{1,2} = \frac{\theta \pm \sqrt{\theta^2 - 4\theta}}{2} = \frac{\theta}{2}(1 \pm \sqrt{1 - \frac{4}{\theta}})$$

The following theorems are the summary of the results obtained here.

Theorem 1.

(1) If $\alpha \epsilon (-1, \frac{-3}{4}]$, then $\overline{x}$ is unstable.

(2) If $\alpha \epsilon (-2, -1)$, then $\overline{x}$ is unstable.

(3) If $\alpha > 0$, then $\overline{x}$ is asymptotically stable.

Proof:

(1) Let $\alpha \in \left(-1, -\frac{3}{4}\right]$, which is $-1 < \alpha \le \frac{-3}{4}$. From $\theta = \frac{1}{\alpha+1}$, we have $0 < \frac{1}{\theta} \le \frac{1}{4}$

Therefore

$$2 > 1 + \sqrt{1 - \frac{4}{\theta}} \ge 1. \quad \frac{\theta}{2} \ge 2$$

Hence

$$\frac{\theta}{2}(1 + \sqrt{1 - \frac{4}{\theta}}) \ge 2$$

In this case, we have $|\lambda_1| > 1$. Therefore $\overline{x}$ is unstable.

(2) Let $\alpha \epsilon (-2, -1)$ which is $-2 < \alpha < -1$

Then

$$1 > -(\alpha + 1) > 0$$

We get $-\theta > 1$ and $0 < -\frac{4}{\theta} < 4$

Then

$$2 < 1 + \sqrt{1 - \frac{4}{\theta}} < \sqrt{5} + 1$$

Hence

$$1 + \sqrt{1 - \frac{4}{\theta}} > 2$$

But we have

$$\frac{-\theta}{2} > \frac{1}{2}$$

Then

$$\frac{-\theta}{2}(1 + \sqrt{1 - \frac{4}{\theta}}) > 1$$

In this case, we have

$$|\lambda_2| = \left|\frac{\theta}{2}(1 + \sqrt{1 - \frac{4}{\theta}}\right| > 1$$

Therefore , $\bar{x}$ is unstable.

(3) Let $\alpha > 0$, we get

$$0 < \theta < 1$$

Let

$$p = -\theta \quad and \quad q = \theta, \theta \epsilon (0,1)$$

where

$$|p| = |-\theta| = \theta < 1 \text{ and } 1 + q = 1 + \theta > 1,$$

we get

$$|p| < 1 + q$$

and

$$q = \theta \in (0,1)$$

This means $q < 1$. Therefore, $\overline{x} = \alpha + 1$ is a asymptotically stable.

Global attractivity

In this section, we study the global attractivity of the equilibrium point where $\overline{x} = \alpha + 1$ of the recursive sequence

$$x_{n+1} = \alpha + \frac{x_n}{x_{n-1}}, n = 0,1, \dots$$

Suppose that there exist $m, M \epsilon (0,1)$ such that α satisfies the following condition

$$\frac{-m}{M(1-M)} \le \alpha \le \frac{-M}{m(1-m)} \qquad (1.4)$$

Lemma 1. Suppose that there exist $m, M \epsilon (0,1)$ such that the condition (1.4) holds. Let $\{x_n\}$ be a solution of equation (1.1). If $x_{n0}, x_{n0-1} \epsilon [\alpha M, \alpha m]$ for some $n_0 \ge 0$. Then,

$$x_{n0+k} \epsilon [\alpha M, \alpha m] \quad \text{for every } k \epsilon N.$$

Proof: Since

$$x_{n0}, x_{n0-1} \epsilon [\alpha M, \alpha m]$$

Then

$$\alpha\, x_{n0-1} \le \alpha^2 M \qquad \text{and} \qquad \frac{x_{n0}}{1-M} \le \frac{\alpha m}{1-M}$$

Hence

$$\alpha\, x_{n_{0-1}} + \frac{x_{n_0}}{1-M} \le \alpha^2 M + \frac{\alpha m}{1-M}$$

By (1.4), we get

$$\alpha\, x_{n_{0-1}} + \frac{x_{n_0}}{1-M} \le 0$$

Hence

$$(1-M)\alpha + \frac{x_{n_0}}{x_{n_{0-1}}} \ge 0 \Longrightarrow \alpha + \frac{x_{n_0}}{x_{n_{0-1}}} \ge \alpha M$$

What's follows

$$x_{n0+1} \ge \alpha M.$$

Now, we will show that

$$x_{n0+1} \le \alpha m.$$

Since

$$x_{n_0}, x_{n_{0-1}} \in [\alpha M, \alpha m]$$

Then

$$\alpha\, x_{n_{0-1}} \ge \alpha^2 m \quad \text{and} \quad \frac{x_{n_0}}{1-m} \ge \frac{\alpha M}{1-m}$$

Then

$$\alpha\, x_{n_{0-1}} + \frac{x_{n_0}}{1-m} \ge \alpha^2 m + \frac{\alpha M}{1-m}$$

By (1.4), we get

$$\alpha\, x_{n_{0-1}} + \frac{x_{n_0}}{1-m} \ge 0$$

Hence

$$(1-m)\,\alpha+\frac{x_{n_0}}{x_{n_{0-1}}}\leq 0\Longrightarrow\ \alpha+\frac{x_{n_0}}{x_{n_{0-1}}}\leq \alpha\, m$$

So,

$$x_{n_{0+1}}\leq\ \alpha\, m$$

Now, let the relation be true when $k=r$, which means that

$$x_{n_{0+r}}\epsilon[\alpha M,\alpha m]$$

We will proof that

$$x_{n_{0+r+1}}\epsilon[\alpha M,\alpha m]$$

Since

$$x_{n_{0+r}}.x_{n_{0+r-1}}\ \epsilon[\alpha M,\alpha m]$$

Then

$$x_{n_{0+r-1}}\geq aM$$

Hence

$$\alpha\, x_{n_{0+r-1}}\leq \alpha^2M \quad \text{and} \quad \frac{x_{n_{0+r}}}{1-M}\leq\frac{\alpha m}{1-M}$$

There fore

$$\alpha\, x_{n_{0+r-1}}+\frac{x_{n_{0+r}}}{1-M}\leq \alpha^2M+\frac{\alpha m}{1-M}$$

By (1.4), we get

$$\alpha\, x_{n_{0+r-1}}+\frac{x_{n_{0+r}}}{1-M}\leq 0$$

Hence

$$(1-M)\alpha+\frac{x_{n_{0+r}}}{x_{n_{0+r-1}}}\geq 0$$

Thus

$$\alpha+\frac{x_{n_{0+r}}}{x_{n_{0+r-1}}}\geq \alpha M.$$

So,

$$x_{n_{0+r+1}}\geq \alpha M$$

Similarly, we can see that $x_{n_{0+r+1}}\leq \alpha m$

Since $x_{n_{0+r}}, x_{n_{0+r-1}}\ \epsilon\ [\alpha M,\alpha m]$, Then we obtain

$$\alpha\, x_{n_{0+r-1}}\geq \alpha^2\text{m} \quad \text{and} \quad \frac{x_{n_{0+r}}}{1-m}\geq\frac{\alpha M}{1-m}$$

So,

$$\alpha\, x_{n_{0+r-1}}+\frac{x_{n_{0+r}}}{1-m}\geq \alpha^2m+\frac{\alpha M}{1-m}$$

By (1.4), we get

$$\alpha\, x_{n_{0+r-1}}+\frac{x_{n_{0+r}}}{1-m}\geq 0$$

Hence,

$$(1-m)\alpha+\frac{x_{n_{0+r}}}{x_{n_{0+r-1}}}\leq 0$$

Hence

$$x_{n_{0+r+1}}\leq \alpha m$$

Which leads to

$$x_{n_{0+k}}\ \epsilon\ [\alpha M,\alpha m] \text{ for every } k\ \epsilon\ N.$$

Now, we can state and prove the following theorem:

Theorem 2. Let $m, M \epsilon (\frac{2}{3}, 1)$ If there exist $m \leq M$ and

$$\frac{-m}{M(1-M)} \leq \alpha \leq \frac{-M}{m(1-m)},$$

Then, $\bar{x}$ is a global attractor with basin $[\alpha M, \alpha m]^2$.

Proof: Let $\{x_n\}$ be a solution with initial conditions

$$x_0, x_{-1} \epsilon [\alpha M, \alpha m]$$

By Lemma 1, we obtain $x_n \, \epsilon \, [\alpha M, \alpha m] \forall \, n \, \epsilon \, N$

Suppose that

$$\lim_{n\to\infty} \inf x_n = \lambda \text{ and } \lim_{n\to\infty} \sup x_n = \mu$$

Let $\varepsilon > 0$ such that $\mu + \varepsilon < 0$

Then, there exists $n_0 \in N$, such that

$$\lambda - \varepsilon < x_n < \mu + \varepsilon$$

Hence

$$-(\lambda - \varepsilon) > -x_n > -(\mu + \varepsilon) \forall n \geq n_0$$

Then

$$-(\lambda - \varepsilon) > -x_{n-1} > -(\mu + \varepsilon) \forall n \geq n_0 + 1$$

So,

$$-\frac{1}{\lambda - \varepsilon} < -\frac{1}{x_{n-1}} < -\frac{1}{\mu + \varepsilon}$$

Then

$$\frac{-1}{\mu + \varepsilon} > \frac{-1}{x_{n-1}} > \frac{-1}{\lambda - \varepsilon} \forall n \geq n_0 + 1$$

Then

$$\frac{\lambda - \varepsilon}{\mu + \varepsilon} > \frac{x_n}{x_{n-1}} > \frac{\mu + \varepsilon}{\lambda - \varepsilon} \quad \forall n \geq n_0 + 1$$

Which means

$$\alpha + \frac{\mu + \varepsilon}{\lambda - \varepsilon} < \alpha + \frac{x_n}{x_{n-1}} < \alpha + \frac{\lambda - \varepsilon}{\mu + \varepsilon} \quad \forall n \geq n_0 + 1$$

Then

$$\alpha + \frac{\mu + \varepsilon}{\lambda - \varepsilon} < x_{n+1} < \alpha + \frac{\lambda - \varepsilon}{\mu + \varepsilon} \quad \forall n \geq n_0 + 1$$

Thus, we get the inequality

$$\alpha + \frac{\mu + \varepsilon}{\lambda - \varepsilon} \leq \lambda \leq \mu \leq \alpha + \frac{\lambda - \varepsilon}{\mu + \varepsilon}$$

This inequality yields

$$\alpha + \frac{\mu}{\lambda} \leq \lambda \leq \mu \leq \alpha + \frac{\lambda}{\mu}$$

Which implies that

$$-\alpha - \frac{\lambda}{\mu} \leq -\lambda \leq -\alpha - \frac{\mu}{\lambda}$$

Then

$$\mu - \lambda \leq \frac{\lambda}{\mu} - \frac{\mu}{\lambda}$$

So, we obtain

$$\mu - \lambda \leq \frac{\lambda^2 - \mu^2}{\mu\lambda}$$

Now,

$$\mu - \lambda > 0$$

Then

$$1 \leq -\frac{\lambda + \mu}{\mu\lambda}$$

Hence

$$\mu\lambda + \mu + \lambda \leq 0$$

On the other hand, the inequality

$$\alpha M \leq \lambda \leq \mu \leq \alpha m$$

Implies the inequality $\alpha^2 m^2 + 2\alpha M \leq \mu\lambda + \lambda + \mu$

Then

$$\alpha^2 m^2 + 2\alpha M \leq 0$$

If $m = \frac{3}{4}$ & $M = \frac{5}{6}$ Then $\frac{-27}{5} \leq \alpha \leq \frac{-40}{9}$

We get $\alpha = -4$

Hence

$$\alpha^2 m^2 + 2\alpha M = 16 \times \frac{9}{16} + 2 \times -4 \times \frac{5}{6}$$

$$= 9 - \frac{20}{3} > 0$$

Which is a contradiction.

Therefore

$$\liminf_{n \to \infty} x_n = \limsup_{n \to \infty} x_n = \bar{x}$$

Then, $\bar{x}$ is a global attractor will basin $[\alpha M, \alpha m]^2$.

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Conflict of interest

The author declare that there are no conflicts of interest.

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