Spatio-Temporal Patterns In Toroidally Coupled Oscillator Systems

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ABSTRACT. This paper presents a study of the spatio-temporal patterns of oscillations that are possible in systems of identical oscillators. These oscillators are symmetrically coupled with the symmetry of a discrete torus. The analysis deals with periodic motions of the entire array rather than individual cells. It exploits the symmetry of the array using results from equivariant bifurcation theory. This work presents a complete list of invariants, equivariants, normal forms, isotropy subgroups and fixed-points subspaces, for the cases with periodicity $N = 2$. It is carried out for the case of a rectangular array with toroidal symmetry. The analysis included all the generic equivariant Hopf bifurcations in this setting and determines the onset, stability and the generic behavior of spatio-temporal patterns for all primary branches. We also find all possible secondary patterns of oscillations using the H/K Theorem.

KEY WORDS. Spatio-temporal pattern formation, coupled oscillator systems, Symmetry breaking, Hopf bifurcation.
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الأنماط ذات الحيز الزمني الموقت للأنظمة الاهتزازية المقترنة

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المستخلص: في هذه الورقة العلمية قمنا بتطبيق نظريه هوف للتفرع للأنظمة المتماثلة وكذلك نظريه H/K للانظمه الاهتزازات المقترنة تحت تأثير زمره التماثل Gn وبشكل خاص على النموذج (n,2) من الحلقات.

من خلال هذا التطبيق تحصلنا على تنوع كاف من وفي هذه الورقة العلمية وعندما تكون n=2 يتبين لنا عند استخدام نظرية التفرع المتماثل فان لكل حل يمكن تعيين زمرة جزئية وكذلك فضاء جزئي ثابت ويكون الحل لهذا الفضاء دوارا وتحت تأثير هذه الزمرة الجزئية ويمكن ان يتمدد هذا التطبيق ويدرس الحالات العامه tori -(n,m)

1. INTRODUCTION:

The mathematical analysis presented here, of spatio-temporal patterns in two dimensional arrays of regularly spaced oscillators with symmetric nearest neighbor coupling, is motivated by flow-induced vibrations of arrays of parallel circular tubes as found in heat exchangers [2].

Symmetries impose constraints on the trajectories and the equilibria of symmetric systems leading to richer possibilities of behavior; thus, symmetries both complicate and facilitate the analysis. The symmetry of a system is measured by its 'equivariance'. Let [i]

$$\frac{dx}{dt} = f(x) \qquad (1)$$

be a system of differential equations, with a smooth function $f : \mathbb{R}^n \to \mathbb{R}^n$, and consider a compact Lie group Γ acting on $\mathbb{R}^n$. Then we say that f is Γ – equivariant if and only if

$$f(\gamma x) = \gamma f(x) \text{ for all } x \in \mathbb{R}^n, \gamma \in \Gamma \qquad (2)$$

Many physical systems can be represented at least approximately as systems of N coupled identical cells, for which the Lie group Γ is a subgroup of the permutation group S_N and the system is Γ- equivariant, [1], [9], [11].

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In this paper, we explore a general setting in which Γ acts on a finite number of rings of cells coupled as a discrete torus, defined by a system of differential equations (1). We apply the Equivariant Hopf Bifurcation Theorem and the H/K Theorem, emphasizing the cases where the number of rings is two and the number of cells per ring is $N = 2$.

Results such as these have surprisingly diverse applications. It is anticipated that the general results presented here will be useful in the analysis of such diverse applications.

2. ISOTROPY SUBGROUPS AND FIXED-POINT SUBSPACES

It is known that the full symmetry Γ of the system may be "broken" spontaneously to less symmetry; that is, to a proper subgroup of Γ. This involves concepts such as group orbit, isotropy subgroup and the fixed-point space [6], [9], [11].

Let Γ be a compact Lie group acting linearly on a vector space V. The orbit group of a point $x \in V$ is defined as

$$\Gamma_x = \{\gamma \cdot x \mid \gamma \in \Gamma\} \qquad (3)$$

If x is an equilibrium solution of (1), i.e. $f(x) = 0$, then every point in its group orbit is also an equilibrium solution, since by the equivariance condition (2), we have
$f(\gamma \cdot x) = \gamma \cdot f(x) = \gamma \cdot 0 = 0$

We have the following dichotomy: either $\gamma x = x$ for every $\gamma \in \Gamma$,(i.e. x has the full symmetry Γ), or $\gamma x \neq x$ for some $\gamma \in \Gamma$ and γx is a new solution. In this case we sat that γx is conjugate to x.

One can quantify the symmetry of a point $x \in V$ as follows. The symmetry of x is measured by its isotropy subgroup of Γ , defined by

$$\Sigma_x = \{\sigma \in \Gamma \mid \sigma \cdot x = x\} \qquad (4)$$

The equivariance of non-linear mappings forces them to have linear invariant subspaces. These invariant subspaces correspond naturally to certain subgroups of Γ and play an important role in reducing the dimension of the system, to a more easily solved reduced equation. If $\Sigma \subset \Gamma$ is any subgroup. The fixed-point subspace of Σ is defined by

$$Fix(\Sigma) = \{x \in V \mid \sigma \cdot x = x \quad \forall \sigma \in \Sigma\} \qquad (5)$$

which is a doubly invariant subspace, not only invariant under the action of the group Γ, but also invariant under the flow defined by the solutions of (1).

For each isotropy subgroup Σ there is a corresponding fixed-point subspace $Fix(\Sigma)$. A solution branch of the system (1) which breaks the symmetry group Γ, and has its own symmetry Σ, must lie in the corresponding fixed-point subspace $Fix(\Sigma)$. Therefore, to find a solution x with symmetry Σ, it suffices to solve the restricted system $f|_{Fix(\Sigma)} = 0$. Under the equivariance property this reduction to fixed-point subspaces is a powerful tool and is used extensively in the analysis of non-linear symmetric systems.

3. INVARIANT AND EQUIVARIANT NORMAL FORMS

Equivariance conditions are important in determining the normal form of the system. For a compact Lie group Γ, acting on a vector space V, a real-valued smooth function $g : V \to \mathbb{R}$ is invariant under Γ if

$$g(\gamma \cdot x) = g(x) \qquad (6)$$

for every $x \in V$, $\gamma \in \Gamma$. An invariant polynomial is defined in the same way, by restricting g to be a polynomial in this definition. An important result for a given action of Γ is that there is always a finite subset of invariant polynomials $\Lambda = \{u_1, \dots, u_s\}$, 'spanning' the invariant functions in the following sense. Any smooth invariant function g can be written as

$$g(x) = h\big(u_1(x), \dots, u_s(x)\big) \qquad (7)$$

for some smooth function h. The set Λ generates a ring and is called a Hilbert basis of invariants.

Now consider an equivariant mapping $f : V \to V$ as above. The equivariant normal form of f can be introduced as follows. There exists a finite set of Γ-equivariant polynomial maps $v_1, \dots, v_r : V \to V$ such that

$$f(x) = \sum_{k=1}^{r} h_k \left(u_1(x), \dots, u_s(x)\right) \cdot v_k(x) \qquad (8)$$

where $h_k(k = 1, \dots, r)$ and $u_j(j = 1, \dots, s)$ are invariant functions as defined above.

4. THE H/K THEOREM

The goal of this paper is to classify all possible spatio-temporal patterns of periodic solutions of $(N, 2)$ toroidally coupled oscillator systems. This goal is accomplished by means of the H/K Theorem.

If $x(t)$ is a T-periodic solution of (1), then for any $\gamma \in \Gamma, \gamma x(t)$ is another T-periodic solution of (1). By the uniqueness of solutions, the trajectories of $\gamma x(t)$ and $x(t)$ either are identical or do not intersect. Suppose that the trajectories do not intersect, then uniqueness of solutions implies that there exists $\theta \in \mathrm{R}$ such that

$$\gamma x(t + \theta) = x(t) \qquad (9)$$

In fact, since $x(t)$ is T-periodic, we can always choose $\theta \in [0, T) \subset \mathrm{R}$. From here on in this paper, let us assume that the time t has been rescaled for convenience, so that $T = 2\pi$. Then, we may equivalently consider θ as

$$\theta \in \mathrm{R}/2\pi \equiv S^1 \qquad (10)$$

Now, the group has many representations, the most intuitive of which is a simple rotation by an angle of $\theta \in [0, 2\pi)$ around a unit circle. In this paper, we consider two different representations of S^1. In the first, θ is an equivalence class of real numbers as in (10), representing addition of θ to the time $t\ (mod\ 2\pi)$. In the second, θ represents an operator on a solution $x(t)$, translating that solution by θ along its orbit; that is,

$$\theta \cdot x(t) = x(t + \theta)$$

Obviously, these two meanings of θ are equivalent here, so there should be no confusion in using the same symbol for both.

Thus, the solution in (9) has a symmetry $(\gamma, \theta) \in \Gamma \times S^1$ that is a mixture of spatial and temporal symmetry. Note that the group S^1 acts on the space of 2π-periodic solutions $x(t)$ and not on R^n. The group of all spatial-temporal symmetries of $x(t)$ in (9) is denoted by

$$\Sigma_{x(t)} = \{ (\gamma, \theta) \in \Gamma \times S^1 \mid (\gamma, \theta) \cdot x(t) = x(t),\ \forall t\} \qquad (11)$$

The subgroup $\Sigma_{x(t)}$ can be identified with a pair of subgroups H and K of Γ and a homomorphism
$\varphi : H \to S^1$ with kernal K. These subgroups are defined as follows:

$$K = \{ \gamma \in \Gamma \mid \gamma x(t) = x(t) \quad \forall t\} \qquad (12)$$
$$H = \{ \gamma \in \Gamma \mid \gamma\{x(t)\} = \{x(t)\}\}$$

The subgroup K is the group of spatial symmetries of $x(t)$ which fixes each point of $x(t)$. The subgroup H consists of those symmetries that preserve the trajectory of $x(t)$; that is, the spatial part of the spatio-temporal symmetries of $x(t)$. The group $\Sigma_{x(t)}$ can be written, for some H and some $\varphi : H \to S^1$, as

$$\Sigma^{\varphi} = \{ (h, \varphi(h)) \in \Gamma \times S^1 \mid h \in H\} \qquad (13)$$

and Σ^{φ} is called a **twisted isotropy subgroup** of $\Gamma \times S^1$. Define

$$L_K = \bigcup_{\gamma \notin K} Fix(\langle\gamma\rangle) \cap fix(K) \qquad (14)$$

Then L_K is the union of proper subspaces of $Fix(K)$. The set of points $Fix(K)/L_K$ has one or more connected components. The necessary and sufficient conditions on the spatial-temporal symmetries for the existence of periodic solutions of a system of (1) are given by the H/K Theorem.

Theorem 4.1. (H/K Theorem) [9]. *Let Γ be a finite group acting on R^n. There is a periodic solution to some Γ-equivariant system of ordinary differential equations on R^n with spatial symmetries K and spatio-temporal symmetries H if and only if*

I. H/K is cyclic.

II. K is an isotropy subgroup.

III. $dim(Fix(K)) \geq 2$. If $dim(Fix(K)) = 2$, then either $H = K$ or $H = N(K)$.

IV. H fixes a connected component of $Fix(K)/L_K$.

5. EQUIVARIANT HOPF BIFURCATION

In this section we review briefly the fundamentals of symmetry breaking as described by the Equivariant Hopf Bifurcation Theorem [9], [11]. We say that an ordinary differential equation

$$\frac{dx}{dt} = f(x,\mu), \quad f(x_0,\mu_0) = 0 \qquad (15)$$

where $f: \mathrm{R}^n \times \mathrm{R} \to \mathrm{R}^n$ is smooth, undergoes a Hopf bifurcation at $\mu = \mu_0$, in the linearized equation of (15) $(df)_{(x_0,\mu_0)}$ has a conjugate pair of purely imaginary simple eigenvalues. With additional hypotheses of non-degeneracy, this condition implies the occurrence of a branch of periodic solutions near (x_0,μ_0). When considering Hopf bifurcation in the presence of symmetry, there is a condition on the action of Γ on the subspace in which $(df)_{(x_0,\mu_0)}$ has purely imaginary eigenvalues. One generically expects the eigenspace to have a special form: that it is 'Γ-simple'.

First, recall that a representation of a group Γ on the subspace W is called irreducible if the only subspaces of W that are invariant under Γ are W and $\{0\}$. Next, define a representation of Γ on W to be absolutely irreducible if the only linear maps on W that commute with Γ are the real multiples of the identity, i.e., $\{cI \mid c \in \mathrm{R}\}$.

Definition 5.1. It is said that the vector space W is Γ-simple if either:

I. $W = V \oplus V$, where the representation of Γ on V is absolutely irreducible or

II. Γ acts irreducibly but not absolutely irreducibly on W.

Therefore, a space W is Γ-simple if it is either non-absolutely irreducible or isomorphic to a direct sum of two copies of the same absolutely irreducible representation. For symmetric systems, "Γ-simple'' is the equivariant analogue of ''simple eigenvalue'' in a generic system. The eigenspace corresponding to the purely imaginary eigenvalues of (df) must be one of the two forms (I) or (II). All the cases of equivariant Hopf bifurcation in the present study are of the form (II).

The Equivariant Hopf Theorem addresses both spatial symmetry and time-periodic solutions; that is, we consider spatio-temporal symmetries $\Gamma \times S^1$ as in the H/K Theorem. Then this theorem asserts the bifurcation of branches of small amplitude periodic solutions of the system (15) with period near 2π, whose spatio-temporal symmetries are those of certain subgroups $\Sigma \subseteq \Gamma \times S^1$. Note that the Hopf Theorem guarantees existence of certain types of periodic solutions, while the H/K Theorem classifies all possible periodic solutions.

Definition 5.2. Let Σ be a twisted isotropy subgroup of $\Gamma \times S^1$ and suppose that $\dim(Fix(K)) = 2$. Then

Σ is called a C- axial isotropy subgroup.

Theorem 5.1. (Equivariant Hopf Bifurcation) [9]. *Let Γ be a compact Lie group acting Γ-simply on R^{2n}. Assume that*

I. $f: R^{2n} \times R \rightarrow R^{2n}$ *is smooth and equivariant,* $f(x_0, \mu) = 0$ *and* $(df)_{(x_0,\mu_0)}$ *has eigenvalues* $\alpha(\mu) \mp i\beta(\mu)$ *each of multiplicity* n.

II. $\alpha(\mu_0) = 0$ *and* $\beta(\mu_0) > 0$.

III. $\alpha'(\mu_0) \neq 0$ *– the eigenvalue crossing condition.*

IV. $\Sigma \subseteq \Gamma \times S^1$ *is a* C*- axial isotropy subgroup.*

Then there exists a unique smooth branch of small amplitude periodic solutions of system (15) with period near 2π*, emanating from* (x_0, μ_0) *with spatio-temporal symmetries* Σ.

6. GLOBAL AND INTERNAL SYMMETRIES

In this section we give a formal mathematical definition of couples oscillator systems in a finite number of rings with two different types of symmetries: global and internal symmetries. The analysis focuses on cases where the number of rings is two and each ring has the same number of identical oscillators.

By an oscillator we mean a system of ODEs with oscillatory solutions, defined on an Euclidean space, for example R^k with $k \geq 2$. (If $k > 2$, often tools such as the center manifold theorem may be used to reduce the dimension to $k = 2$). We define a ring of coupled oscillators to be a coupled oscillator system of the form

$$\frac{dx_i}{dt} = f(x_i) + \sum_{j=1}^{N} h_{ij}(x_i, x_j), \qquad i = 1, \dots, N \qquad (16)$$

with either $\mathbb{Z}_N$ or $\boldsymbol{D}_N$ symmetry of h_{ij}. It is a unidirectional ring if the oscillators are coupled with $\mathbb{Z}_N$- symmetry, and bidirectional if the oscillators are coupled with $\boldsymbol{D}_N$-symmetry. To generalize the form (16) for a finite number of rings of coupled oscillators, suppose we have M rings each of which has N identical oscillators. Then the coupled identical oscillators may be represented by a system of differential equations of the form

$$\frac{dx_{ij}}{dt} = f(x_{ij}) + \sum_{q=1}^{N} h_{ijq}(x_{ij}, x_{iq}) + \sum_{p=1}^{M} g_{pij}(x_{ij}, x_{pj}) \qquad (17)$$

where $i = 1, \dots, M,\ j = 1, \dots, N,\ x_{ij}$ is the state vector of the ij^{th} oscillator , $f: R^k \rightarrow R^k$ represents the internal dynamics of each oscillator, $h_{ijq}: R^k \times R^k \rightarrow R^k$ is the coupling function in the M rings, and $g_{pij}: \mathrm{R}^k \times \mathrm{R}^k \rightarrow \mathrm{R}^k$ is the coupling function in the N rings. For simple notation, let (N, M) denote M rings of N identical oscillators, or equivalently N rings each of which has identical oscillators. Such a system may be called a discrete (N, M)-torus of coupled oscillators. The state space of the entire coupled system of (N, M)-oscillators is then

$$X^* = (\mathrm{R}^k)^N \times (\mathrm{R}^k)^N \times \dots \times (\mathrm{R}^k)^N \quad (M\ times) \qquad (18)$$

For the purpose of this paper we assume that the rings are unidirectionally coupled, that is, with $\mathbb{Z}_N$- symmetry.

The key idea we wish to explore here is the spatio-temporal symmetries of solutions of unidirectional (N, M)-tori of coupled oscillators. Since we may have M identical rings, each of N identical oscillators, the global symmetry group is

$$\Gamma = \mathbb{Z}_N \times \mathbb{Z}_M \qquad (19)$$

The group element $(\rho, \kappa) \in \Gamma$ where $\rho \in \mathbb{Z}_N$ and $\kappa \in \mathbb{Z}_M$ can be written as a single element $\rho\kappa \in \Gamma$. Note that $\rho\kappa = \kappa\rho$, that is, Γ is abelian. Now assume that the number of rings is 2 $(M = 2)$, each of which has identical oscillators. In this case the actions of ρ and κ are defined to be $\rho \cdot x_{i,j} = x_{i,j+1}$ and $\kappa \cdot x_{i,j} = x_{i+1,j}$ where i is taken modulo 2 and j is taken modulo N. Then the global symmetry of the $(N, 2)$-coupled oscillator system is

$$\mathrm{G}_N = \mathbb{Z}_N(\rho) \times \mathbb{Z}_2(\kappa) \qquad (20)$$

$$= \{(1,1), (\rho, 1), \dots, (\rho^{N-1}, 1), (1, \kappa), (\rho, \kappa), \dots, (\rho^{N-1}, \kappa)\}$$

which can be written more compactly as

$$\mathbb{G}_N = \{1, \rho, \rho^2, \dots, \rho^{N-1}, \kappa, \rho\kappa, \dots, \rho^{N-1}\kappa\}$$

where the index N refers to the number of oscillators in each ring. Note that $\mathbb{G}_N$ is an abelian group with $2N$ elements.

The local internal symmetry group of the (N, M) -coupled oscillators can be described in the following way. Let Ω be a subgroup of $O(k)$, that is the k dimensional orthogonal group

$$\boldsymbol{O}(k) = \{A \in \boldsymbol{GL}(k, \mathbb{R}) \mid A^T A = I_k\}$$

For Ω to be an internal symmetry group, we require first that every $\sigma \in \Omega$ must satisfy the following condition

$$f(\sigma \cdot x_{ij}) = \sigma \cdot f(x_{ij}) \qquad (21)$$

Here $f(x_{ij})$ defines the internal dynamics of the ij^{th} oscillator. These internal symmetries are symmetries of the entire (N, M)-torus of coupled oscillators provided certain properties of the coupling functions h_{ijq} and g_{pij} hold. We require that each σ acts simultaneously on every cell. Then it is a symmetry of the coupled oscillator system if the following conditions are satisfied for all i, j, p, q

$$h_{ijq}(\sigma \cdot x_{iq}, \sigma \cdot x_{ij}) = \sigma \cdot h_{ijq}(x_{iq}, x_{ij}) \qquad (22)$$

$$g_{pij}(\sigma \cdot x_{pj}, \sigma \cdot x_{ij}) = \sigma \cdot g_{pij}(x_{pj}, x_{ij})$$

If we define $\boldsymbol{x}^* \in X^*$ and

$$\sigma \cdot \boldsymbol{x}^* = (\sigma x_{i1}, \sigma x_{i2}, \dots, \sigma x_{iN}), \qquad i = 1,2, \dots, M$$

and define F by writing equation (17) as

$$\frac{dX^*}{dt} = F(X^*)$$

then the internal symmetry condition is that F is Ω-equivariant, that is

$$F(\sigma \cdot \boldsymbol{x}^*) = \sigma \cdot F(\boldsymbol{x}^*) \qquad \forall\, \sigma \in \Omega, \boldsymbol{x}^* \in X^* \qquad (23)$$

It follows that all $(\sigma, \gamma) \in \Omega \times \mathbb{G}_N$ are symmetries of (17). This type of combination of internal and global (external) symmetries is known as *direct product coupling* [7], [8].

7. POSSIBLE PERIODIC SOLUTIONS

This section presents a characterization of all possible periodic solutions, their twisted isotropy subgroups and corresponding fixed-point subspaces. It is obtained by applying the H/K Theorem to the systems defined above. The results are presented here in a series of tables.

There are several good reasons to assume that the state space of each physical oscillator in the system has dimension $k = 2$. Topologically, $k = 2$ is the minimum dimension for a limit cycle oscillator to exist. Under the hypotheses of the Hopf bifurcation theorem, there exists an invariant 2-dimensional center manifold in which the bifurcating limit cycle exists. Near the Hopf bifurcation point, generic center manifold arguments reduce the system to a 2-dimensional ODE. We can assume that this reduction has been performed for our equations.

From (18) the state space of a $(N, 2)$-torus of coupled oscillators where each oscillator is considered to be two dimensional is

$$X^* = (\mathbb{R}^2)^N \times (\mathbb{R}^2)^N$$
$$= \mathbb{R}^{2N} \times \mathbb{R}^{2N}$$

Equivalently, the state space can be considered to be a product of two N-dimensional complex spaces using natural isomorphism of $\mathbb{R}^2$ with C. Let the state space of the $(N, 2)$-coupled oscillator system be represented by

$V = \mathbb{C}^N \times \mathbb{C}^N$

The natural action of the group $\mathbb{G}_N$ on V is given by

$$\rho \cdot ((z_1, z_2, \dots, z_N), (w_1, w_2, \dots, w_N)) = ((z_N, z_1, \dots, z_{N-1}), (w_N, w_1, \dots, w_{N-1}))$$
$$\kappa \cdot ((z_1, z_2, \dots, z_N), (w_1, w_2, \dots, w_N)) = ((w_1, w_2, \dots, w_N), (z_1, z_2, \dots, z_N)) \quad (24)$$

The goal of this section is to apply the H/K Theorem for the case when $N = 2$, so that we can find the periodic solutions.

When $N = 2$, the state space of the $(2,2)$-coupled oscillator system is $V = \mathbb{C}^2 \times \mathbb{C}^2$ and the external symmetry group may be written as

$\mathbb{G}_2 = \{1, \kappa_1, \kappa_2, \kappa_1\kappa_2\}$

which acts on V as follows

$$\kappa_1 \cdot ((z_1, z_2), (w_1, w_2)) = ((z_2, z_1), (w_2, w_1))$$
$$\kappa_2 \cdot ((z_1, z_2), (w_1, w_2)) = ((w_1, w_2), (z_1, z_2))$$

All the possible symmetries of periodic solutions for the case of $N = 2$, as follows from the H/K Theorem, are listed in Table 1, where $\pi \in S^1$ denotes the half-period phase shift, written additively. The twisted isotropy subgroups with their corresponding fixed-point subspaces and dimensions are in Table 2.

Table 1. Periodic solutions of a $\mathbb{G}_2$-equivariant system from the H/K Theorem.

(H, K) Type	Corresponding Periodic Solution
$(\mathbb{G}_2, \mathbb{G}_2)$	$((z(t), z(t)), (z(t), z(t)))$
$(\mathbb{G}_2, \mathbb{Z}_2(\kappa_1))$	$((z(t), z(t)), (z(t+\pi), z(t+\pi)))$
$(\mathbb{G}_2, \mathbb{Z}_2(\kappa_2))$	$((z(t), z(t+\pi)), (z(t), z(t+\pi)))$
$(\mathbb{G}_2, \mathbb{Z}_2(\kappa_1\kappa_2))$	$((z(t), z(t+\pi)), (z(t+\pi), z(t)))$
$(\mathbb{Z}_2(\kappa_1)\mathbb{Z}_2(\kappa_1))$	$((z(t), z(t)), (w(t), w(t)))$
$(\mathbb{Z}_2(\kappa_1), 1)$	$((z(t), z(t+\pi)), (w(t), w(t+\pi)))$
$(\mathbb{Z}_2(\kappa_2), \mathbb{Z}_2(\kappa_2))$	$((z(t), w(t)), (z(t), w(t)))$
$(\mathbb{Z}_2(\kappa_2), 1)$	$((z(t), w(t)), (z(t+\pi), w(t+\pi)))$
$(\mathbb{Z}_2(\kappa_1\kappa_2), \mathbb{Z}_2(\kappa_1\kappa_2))$	$((z(t), w(t)), (w(t), z(t)))$
$(\mathbb{Z}_2(\kappa_1\kappa_2), 1)$	$((z(t), z(t)), (w(t+\pi), z(t+\pi)))$
$(1,1)$	$((z_1(t), z_2(t)), (w_1(t), w_2(t)))$

Table 2. The twisted isotropy subgroups with their corresponding fixed-point subspaces

and dimensions, for a $\mathbb{G}_2$-equivariant system.

Twisted Isotropy Subgroup Σ	Fixed-Point Subspace $Fix(\Sigma)$	dim $Fix(\Sigma)$
$\mathbb{G}_2 \times \{1\}$	$\{((z, z), (z, z))\}$	2
$\mathbb{G}_2(\kappa_1, \kappa_2\pi)$	$\{((z, z), (z+\pi, z+\pi))\}$	2
$\mathbb{G}_2(\kappa_1\pi, \kappa_2)$	$\{((z, z+\pi), (z, z+\pi))\}$	2
$\mathbb{G}_2(\kappa_1\pi, \kappa_2\pi)$	$\{((z, z+\pi), (z+\pi, z))\}$	2
$\mathbb{Z}_2(\kappa_1)$	$\{((z, z), (w, w))\}$	4
$\mathbb{Z}_2(\kappa_1\pi)$	$\{((z, z+\pi), (w, w+\pi))\}$	4
$\mathbb{Z}_2(\kappa_2)$	$\{((z, w), (z, w))\}$	4
$\mathbb{Z}_2(\kappa_2\pi)$	$\{((z, w+\pi), (z+\pi, w))\}$	4
$\mathbb{Z}_2(\kappa_1\kappa_2)$	$\{((z, w), (w, z))\}$	4
$\mathbb{Z}_2(\kappa_1\kappa_2\pi)$	$\{((z, w), (w+\pi, z+\pi))\}$	4
I	$\{((z_1, z_2), (w_1, w_2))\}$	8

8. $\mathbb{G}_N$ - IRREDUCIBLE REPRESNTATIONS

The goal of this section is to discuss irreducible and absolutely irreducible actions of the group $\mathbb{G}_N$ on the space $V = \mathbb{C}^2 \times \mathbb{C}^2$.

According to the results in Table 2, the action of the group $\mathbb{G}_N$ on V is not irreducible. However, it is also not absolutely irreducible because of the relation $\rho\kappa = \kappa\rho$ in the definition of the group $\mathbb{G}_N$. That is, the linear mappings ρ and κ commute; that is what makes the condition of absolute irreducibility fail.

First, decompose V into the following direct sum of invariant subspaces

$$V = D_1 \oplus D_2 \oplus D_0$$

where

$$D_1 = \{(z, z, \dots, z), (z, z, \dots, z) \mid z \in \mathbb{C}\}$$
$$D_2 = \{(z, z, \dots z), (-z, -z, \dots, -z) \mid z \in \mathbb{C}\}$$
$$D_0 = \{((z_1, z_2, \dots, z_N), (w_1, w_2, \dots, w_N)) \mid z_i, w_j \in \mathbb{C}, \quad \Sigma z_i = 0, \Sigma w_j = 0,\}$$

It is clear that the actions of the group $\mathbb{G}_N$ on D_1 and D_2 are both irreducible. However, the action on D_0 is not irreducible, because there are invariant subspaces such as

$$\{((z, wz, \dots, w^{N-1}z), (z, wz, \dots, w^{N-1}z)) \mid z \in \mathbb{C}\}$$

of D_0, which are neither $\{0\}$ on C_0. Here we let ω be any N^{th} complex root of unity in $\mathbb{C}$; then $1 + \omega + \omega^2 + \cdots + \omega^{N-1} = 0$.

By the Theorem of Complete Reducibility, we can decompose the space V into a finite number of $\mathbb{G}_N$-irreducible subspaces, $D_1, D_2, \dots, D_{2N}$, such that

$$V = D_1 \oplus D_2 \oplus D_3 \oplus \dots \oplus D_{2N} \qquad (25)$$

where D_1, D_2 are as above and each D_j ,$j = 3, \dots, 2N$ is a proper subspace of D_0.

Theorem 8.1. $\mathbb{G}_N$ *acts non-absolutely irreducibly on each* D_j ,$j = 1, \dots, 2N$.

Proof. For each case define a linear mapping on V with matrix representation $M = \delta\kappa$ where $0 \neq \delta \in \mathbb{R}$, and

$$\kappa = \begin{pmatrix} 0 & I_N \\ I_N & 0 \end{pmatrix}$$

The matrix M in this choice is not a scalar multiple of the identity but it commutes with ρ and $\kappa \in \mathbb{G}_N$. Therefore, $\mathbb{G}_N$ acts on D_j ,$j = 3, \dots, 2N$, non-absolutely irreducibly.

Thus, each subspace in the decomposition (25) is irreducible but not absolutely irreducible, and hence is $\mathbb{G}_N$-simple. This decomposition is also the isotypic decomposition of the entire space V, because none of these subspaces is $\mathbb{G}_N$- isomorphic to any other. It is now possible to apply the Equivariant Hopf Bifurcation Theorem in each of the subspaces (25) where $\mathbb{G}_N$ acts nontrivially.

9. INVARIANTS AND EQUIVARIANTS FOR $\mathbb{G}_N \times S^1$

Before we can apply the Equivariant Hopf Bifurcation Theorem, it is necessary to enlarge the group $\mathbb{G}_N$ to a group $\mathbb{G}_N \times S^1$ acting on V and find its invariants and equivariants.
The group S^1 acts naturally on in the standard way as follows

$$\theta \cdot ((z_1, \dots, z_N), (w_1, \dots, z_N)) = ((e^{i\theta} z_1, \dots, e^{i\theta} z_N), (e^{i\theta} w_1, \dots, e^{i\theta} w_N)) \quad (26)$$

where $\theta \in S^1$ and $z_i, w_i \in \mathbb{C}$. note that this is yet another representation of S^1, distinct from those in section 4. All of these representatives are, of course, isomorphic. We may work with whichever is convenient for the purposes at hand.

Consider parametrized equivariant vector fields f where

$$f : V \times \mathbb{R} \to V$$

with V as in (25), and where $\mathbb{G}_N$ acts irreducibly on $D_1, D_2, \dots, D_{2N}$. These subspaces D_j are two-dimensional and each represents a fixed-point subspace $Fix(\Sigma)$ for a certain C-axial subgroup Σ and each can be identified with $\mathbb{C}$. Therefore the $\mathbb{G}_N \times S^1$-equivariance of f implies that

$$f : Fix(\Sigma) \times \mathbb{R} \to Fix(\Sigma)$$

The S^1-invariants and equivariants can be found in general in [11]. Here we use this lemma, where invariants and equivariants are considered on $\mathbb{C}^m$ and where $\theta \in S^1$ acts on $\mathbb{C}$ by $\theta \cdot z = e^{i\theta} z$, but we adapt this Lemma to the spaces $D_j \cong \mathbb{C}$ (that is $m = 1$).

Lemma 9.1. [11]. *I. A Hilbert basis for the S^1-invariant function $g : D_j \cong C \to R$ is given by the polynomial*

$$u = z\bar{z}$$

II. Let $f : C \to C$ be S^1-equivariant, that is satisfies $f(\theta \cdot z) = e^{i\theta} f(z)$ with as in (26). Then the module of f such is generated by the two mappings

$$X(z) = z \quad \text{and} \quad Y(z) = iz$$

Note that the action of $\mathbb{G}_N$ on $\mathbb{C}$ will depend on N and have different representations on different D_j. For example, for $N = 2$, the action of $\mathbb{G}_2$ on $D_1 \cong \mathbb{C}$ is trivial and on $D_2 \cong \mathbb{C}$ is defined (with $e^{i\pi} z = -z$) by

$$\rho \cdot z = z \text{ and } \kappa \cdot z = -z$$

However, on $D_3 = \{((z, -z), (z, -z)) \mid z \in \mathbb{C}\} \cong \mathbb{C}$ the action is defined by

$$\rho \cdot z = -z \text{ and } \kappa \cdot z = z$$

Even so, in all the cases one has the following result.

Proposition 9.1. [11]. I. *Every smooth $\mathbb{G}_N \times S^1$-invariant function $g : D_j \cong \mathbb{C} \to \mathbb{R}$ has the form, where is a smooth real-valued function of the invariant polynomial*

$$u = z\bar{z}$$

II. Every smooth $\mathbb{G}_N \times S^1$-equivariant function $f : D_j \cong C \to C$ has the form

$$f(z) = P(u)z + Q(u)\, iz$$

where P and Q are smooth real-valued functions of the invariant polynomial u as in I.

Now, considering parametrized families of equivariant $f(z, \mu)$, the bifurcation problem can be written as

$$f(z, \mu) = P(u, \mu)z + Q(u, \mu)iz \qquad (27)$$

where the functions P and Q are real and depend only on $u = z\bar{z}$ and the real bifurcation parameter μ.

9.1 BRANCHING EQUATION AND STABILITY ANAYLSIS

As shown above, the nature of the action of the group $\mathbb{G}_N$ forces the symmetric $(N, 2)$-tori of identical oscillators to have $2N$ $\mathbb{G}_N$-simple subspaces $D_j, j = 1, \dots, 2N$. Therefore, we need to analyze $2N$ Hopf bifurcation problems, each of the same form. It is clear that the only difference between the problems for different D_j is in the definition of P_j and Q_j. Therefore, all of the $\mathbb{G}_N$ -equivariant symmetry-breaking Hopf bifurcation are defined by ODE's of the form.

$$\frac{dz}{dt} = P_j(u, \mu)z + Q_j(u, \mu)iz, \quad j = 1, \dots, 2N. \qquad (28)$$

where P and Q are as in Proposition 9.1.

An important property of (28) is that it can be decoupled into phase-amplitude equations. Let

$$z = re^{i\varphi}$$

That implies $r^2 = u = z\bar{z}$. The corresponding amplitude and phase differential equations are both real, and given respectively by

$$\dot{r} = P_j(r^2, \mu)r \qquad (29)$$
$$\dot{\varphi} = Q_j(r^2, \mu)$$

An equilibrium of the amplitude equation corresponds to either an equilibrium point $(r = 0)$ or a periodic orbit $(r > 0)$ in the system (28). Also, the stability of an equilibrium of the amplitude equation in (29) is the same as the stability of the corresponding equilibrium or periodic solutions of (28). We have the following possible solutions:

I. $r = 0$, a trivial solution or $\mathbb{G}_N$ -invariant equilibrium.

II. $P_j(r^2, \mu) = 0$, giving a periodic solution of amplitude r.

In case of II, $P_j(r^2, \mu) = 0$ is the branching equation. Also, we need to ensure that $\dot{\varphi} = Q_j(r^2, \mu_j) \neq 0$, which determines the angular rotation speed. Altogether, a non-degenerate Hopf bifurcation requires the following conditions:

$$P_j(0, \mu_j) = 0, \quad P_{j\mu}(0, \mu_j) \neq 0, \quad Q_j(0, \mu_j) \neq 0$$

where μ_j is the bifurcation point, corresponding to the fixed-point subspace D_j . The eigenvalue ‘’crossing condition’’ $P_{j\mu}(0, \mu_j) \neq 0$ guarantees that there is a solution of the branching equation. To see this, let the bifurcation point $\mu_j(0) \equiv \mu_j$ be given. Then if $P_{j\mu}(0, \mu_0) \neq 0$ for sufficiently small u , there exists a unique branch of solutions $u = \mu_j(u)$ satisfying $P_j(u, \mu) = 0$ and $u = \mu_j(u)$ by the Implicit Function Theorem.

The two derivatives $P_{j\mu}(0, \mu_0) \neq 0$ and $P_{j\mu}(0, \mu_0) \neq 0$ can be used to determine whether a solution branch is subcritical or supercritical. Using Taylor’s expansion about the point $(0, \mu_0)$ we obtain

$$P_j(u, \mu) = P_j(0, \mu_0) + (\mu - \mu_0)P_{j\mu}(0, \mu_0) + uP_{j\mu}(0, \mu_0) +$$

Since $P_j(0, \mu_0) = 0$ then we have

$$(\mu - \mu_0) = -\frac{P_{j\mu}(0, \mu_0)}{P_{j\mu}(0, \mu_0)}u + \cdots$$

If, as is usually the case, $P_{j\mu}(0, \mu_0) > 0$ (the eigenvalues cross the imaginary axis from left to right as μ increases), then it follows that the solution branch is supercritical when $P_{j\mu}(0, \mu_0) < 0$ and subcritical when $P_{j\mu}(0, \mu_0) > 0$.

The stability of the branch is guaranteed by the Hopf Theorem under the condition that the trivial solution is originally stable. If $\mu > \mu_0$ the trivial solution becomes unstable and by the ‘’Exchange of Stabilities” the supercritical branch will be a stable.

Next, we compute the stability of branches of solutions for each C-axial twisted isotropy subgroup Σ_j using the signs of the real parts of eigenvalues of the Jacobian matrix. For each $D_j \cong \mathbb{C}$, where $j = 1, \ldots, 2N$, in the complex coordinates $(z, \bar{z})$, recall that the general R-linear mapping on $\mathbb{C} \cong \mathbb{R}^2$ has the form

$$w \to \alpha w + \beta \bar{w}$$
$$\bar{w} \to \bar{\beta} w + \bar{\alpha} \bar{w} \qquad (30)$$

where $\alpha, \beta \in \mathbb{C}$. A simple calculation shows that

$$trace = 2Re(\alpha) \text{ and } det = |\alpha|^2 - |\beta|^2$$

Let λ_{j1} and $\lambda_{j2} = \overline{\lambda_{j1}}$ be the complex conjugate eigenvalues for each D_j. From (30) we have

$$dh_j(w) = \alpha_j w + \beta_j \bar{w}$$

where $\alpha_j \equiv h_{jz}$ and $\beta_j \equiv h_{j\bar{z}}$. Now

$$h_{jz} = P_j(u) + (P_{ju}u_z)z + iQ_j(u) + (iQ_{ju}u_z)z$$

$$h_{j\bar{z}} = (P_{ju}u_{\bar{z}})z + (iQ_{ju}u_{\bar{z}})z$$

Then evaluating h_{jz} and $h_{j\bar{z}}$ at $(0,\mu)$ we obtain

$$\alpha_j = P_j(0,\mu) + iQ_j(o,\mu)$$

$$\beta_j = 0$$

Therefore, for each j the complex conjugate eigenvalues of dh_j are given by

$$trace = 2P_j(0,\mu_j)$$

$$det = \alpha\bar{\alpha} = |P_j(0,\mu_j)|^2 - |Q_j(0,\mu_j)|^2 > 0$$

that is, for each j the complex conjugate eigenvalues of dh_j are given by

$$\lambda_j{}^2 - 2[P_j(0,\mu)]\lambda_j + \left[|P_j(0,\mu)|^2 + |Q_j(0,\mu)|^2\right] = 0$$

It is clear that at the bifurcation point $\mu = \mu_0$ we have $\alpha = iQ_j(0,\mu_j)$ and the trace is 0, and hence the eigenvalues λ_{j1} and λ_{j2} are a pair of purely imaginary eigenvalues. For $\mu > \mu_0$, if $P_{j\mu}(0,\mu_j) > 0$ then the trivial equilibrium solution branch is unstable, and it is stable if $P_{j\mu} < 0$.

10. HOPF BIFURCATION WITH $G_N \times S^1$-SYMMETRY

This section presents the equivariant Hopf bifurcations with $G_N \times S^1$-symmetry, focusing on the case of $N = 2$. The study presents the twisted isotropy subgroups of the group $G_N \times S^1$, together with their fixed points subspaces, and the invariant and equivariant polynomials for this case. The stability is investigated and non-degeneracy conditions corresponding to each solution of the bifurcation problem are determined.

10.1 Case I: G_2-symmetry

The simplest case is equivariant Hopf bifurcation with a global G_2-symmetry; that is a (2,2)-torus of coupled oscillators. The state vector

$$X = ((z_1, z_2), (w_1, w_2))$$

represents the state variables for the entire 4-oscillator system. Under the assumption that the oscillators are identical, with $z_j \in C \cong R^2$ and $w_j \in C \cong R^2$, we have

$$X \in C^2 \times C^2 \cong R^4 \times R^4 \cong R^8$$

The symmetry group of this system is $G_2 = Z_2 \times Z_2$. For this case we write (ρ,κ) as (κ_1,κ_2), so that

$$G_2 = \{1, \kappa_1, \kappa_2, \kappa_1\kappa_2\}$$

where

$Z_2(\kappa_1) = \{1, \kappa_1\}$ $\quad$ $Z_2(\kappa_2) = \{1, \kappa_2\}$

The group G_2 acts on $C^2 \times C^2$ as follows

$$\kappa_1 \cdot ((z_1, z_2), (w_1, w_2)) = ((z_2, z_1), (w_2, w_1)) \tag{31}$$

$$\kappa_2 \cdot ((z_1, z_2), (w_1, w_2)) = ((w_1, w_2), (z_1, z_2))$$

Now consider the action of the group G_2 on each of the irreducible subspaces of $C^2 \times C^2$ defined by

$$D_1 = \{ ((z,z),(z,z)) \mid z \in C\}$$

$$D_2 = \{ ((u,u),(-u,-u)) \mid u \in C\}$$

$$D_3 = \{((v,-v),(v,-v)) \mid v \in \mathbb{C}\}$$
$$D_4 = \{((w,-w),(-w,w)) \mid w \in \mathbb{C}\}$$

It is clear that

$$X = D_1 \oplus D_2 \oplus D_3 \oplus D_4$$

It follows that each j is $\mathbb{G}$-simple. Note that each D_j is two-dimensional and may support equivariant Hopf bifurcation since it could be the eigenspace of a pair of purely imaginary eigenvalues $\mp i\omega$, of a linear operator that commutes with $\mathbb{G}_2$. Extend the action to the group $\mathbb{G}_N \times S^1$ on each subspace D_j, where $\theta \in S^1$ acts on X as in (26). The possible periodic solutions living in these 2-dimensional subspaces are a subset of those found in section 7 using the H/K Theorem and presented in tables 1and 2. Here, only the C-axial symmetry subgroups and their fixed-point subspaces are listed in table 3.

Table 3. The C-axial isotropy subgroups and their corresponding fixed-point subspaces, for $\mathbb{G}_N$-symmetry acting on $\mathbb{C}^2 \times \mathbb{C}^2$

Twisted Isotropy Subgroup	Fixed-Point Subspace $Fix(\Sigma)$	$\dim(Fix(\Sigma))$
$\mathbb{G}_2(\kappa_1,\kappa_2)$	$\{((z,z),(z,z))\}$	2
$\mathbb{G}_2(\kappa_1,\kappa_2\pi)$	$\{((z,z),(z+\pi,z+\pi))\}$	2
$\mathbb{G}_2(\kappa_1\pi,\kappa_2)$	$\{((z,z+\pi),(z,z+\pi))\}$	2
$\mathbb{G}_2(\kappa_1\pi,\kappa_2\pi)$	$\{((z,z+\pi),(z+\pi,z))\}$	2

The periodic solutions corresponding to table 3 are the following

$$\{((z(t),z(t)),(z(t),z(t)))\} \subset D_1$$
$$\{((z(t),z(t)),(z(t+\pi),z(t+\pi)))\} \subset D_2$$
$$\{((z(t),z(t+\pi)),(z(t),z(t+\pi)))\} \subset D_3$$
$$\{((z(t),z(t+\pi)),(z(t+\pi),z(t)))\} \subset D_4$$

Now we analyze the Hopf bifurcation problem for each vector field f_j where

$f_j : D_j \times \mathbb{R} \to D_j$

for each of D_1, D_2, D_3, D_4 with $Fix(\Sigma_j) = D_j$. It follows from (31) that $\mathbb{G}_2$ acts on each $D_j \cong \mathbb{C}$ as in table 4.

Table 4. Action of $\mathbb{G}_2$ on each D_j, $j = 1, \dots, 4$

D_1	D_2	D_3	D_4
$\kappa_1 \cdot z = z$	$\kappa_1 \cdot z = z$	$\kappa_1 \cdot z = z + \pi$	$\kappa_1 \cdot z = z+\pi$
$\kappa_2 \cdot z = z$	$\kappa_2 \cdot z = z + \pi$	$\kappa_2 \cdot z = z$	$\kappa_2 \cdot z = z+\pi$

The invariants and equivariants for $\mathbb{G}_2 \times S^1$ are as follows.

Corollary 10.1. *I. Every smooth $\mathbb{G}_2 \times S^1$-invariant function $g : C \to R$ has the form $P(u)$ where*

$$u = z\bar{z}$$

II. Every smooth $\mathbb{G}_2 \times S^1$ -equivariant function has the form $f : C \to C$

$$f(z) = P(u)z + Q(u)iz$$

with P and Q smooth real functions as in I.

Proof. Directly from Lemma 9.1 and Proposition 9.1.

Therefore we need to analyze four Hopf bifurcation problems of the form

$$\frac{dz}{dt} = P_j(u,\mu)z + Q_j(u,\mu)iz\,, \quad j = 1,\dots,4 \tag{32}$$

where P_j and Q_j are as in Corollary 1 and each equivariant vector filed is defined on the corresponding D_j in tables 3 and 4. The problem on D_1 is a classical Hopf bifurcation and the other are equivariant Hopf bifurcation problems. The only difference between these problems is the definition of the functions P_j and Q_j.

10.2 $\mathbb{G}_2$ Branching Equations and Eigenvalues

The branching equations and the signs of the corresponding eigenvalues of the branching solutions in each $\mathbb{G}_2$-equivariant bifurcation problem are determined by quantities in table 5.

Table 5. The branching equations and the signs of the eigenvalues for $\mathbb{G}_2$- symmetry acting on $D_j, j = 1,\dots,4$

Isotropy Σ	Fixed Σ	Branching Equations	Signs of Eigenvalues
$\mathbb{G}_2(\kappa_1,\kappa_2)$	D_1	$P_1(r^2,\mu) = 0$	Trace = $2P_1(0,\mu_1)$ det= $\lvert P_1(0,\mu_1)\rvert^2 + \lvert Q_1(0,\mu_1)\rvert^2$
$\mathbb{G}_2(\kappa_1,\kappa_2\pi)$	D_2	$P_2(r^2,\mu) = 0$	Trace = $2P_2(0,\mu_2)$ det= $\lvert P_2(0,\mu_2)\rvert^2 + \lvert Q_2(0,\mu_2)\rvert^2$
$\mathbb{G}_2(\kappa_1\pi,\kappa_2)$	D_3	$P_3(r^2,\mu) = 0$	Trace = $2P_3(0,\mu_3)$ det= $\lvert P_3(0,\mu_3)\rvert^2 + \lvert Q_3(0,\mu_3)\rvert^2$
$\mathbb{G}_2(\kappa_1\pi,\kappa_2\pi)$	D_4	$P_4(r^2,\mu) = 0$	Trace = $2P_4(0,\mu_4)$ det= $\lvert P_4(0,\mu_4)\rvert^2 + \lvert Q_4(0,\mu_4)\rvert^2$

The following theorem classifies the non-degeneracy conditions for the stability of each solution, $j = 1,\dots,4$.

Theorem 10.2.1. *Assume simple eigenvalues and isolate bifurcation points μ_j, for each of the isotropy subgroups in table 5. Assume also that $P_{j\mu}(0,\mu_j) > 0$, so that the trivial branch loses stability as μ increases through μ_j. Then, there exists precisely one smooth branch of small amplitude, near 2π-periodic solutions, in a neighborhood of $(0,\mu_j)$. The solution branch in $D_j, j = 1,\dots,4$ is supercritical or subcritical according to whether $P_{j\mu}(0,\mu_j)$ is negative or positive, respectively. At the smallest of the bifurcation point μ_j, a supercritical branch is stable. Any subcritical branch is unstable.*

Proof. Follows directly from the equivariant Hopf bifurcation theorem.

11. CONCLUSIONS

In this paper we have applied the equivariant Hopf bifurcation theorem and the H/K Theorem to coupled systems of identical oscillators with $\mathbb{G}_N$-symmetry, with the symmetries of a discrete $(N,2)$ torus. A rich variety of spatio-temporal patterns is possible in such systems. These patterns and their onset have been identified in this paper for $N = 2$ using equivariant bifurcation theory. For each solution we determine the corresponding twisted isotropy subgroup and the fixed-point subspaces.

Many ideas for further applications and directions for further research have come to light. This work can be extended to (N,M)-tori of coupled identical oscillators. This work can also be extended by assuming different coupling geometries, such as a hexagonal lattice, or by assuming the wreath product coupling of internal symmetries in the model.

Future work shall include cases where $N = 3$ and $N = 4$ also using equivariant Hopf bifurcation theory.

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