Averaging Techniques in The Oscillations of Nonlinear Differential Equations With Alternating Coefficients

Fatima N. Ahmed

Abstract: This paper is concerned with the oscillation theory of a class of second order nonlinear differential equations. Several sufficient conditions for the oscillation of all solutions are established. By using the average property of the integrals of the alternating coefficients as well as the Riccati transformation many oscillatory theorems are proved. However, and as it will be explained, the obtained results here generalize and improve some previous results in the literature. An example is provided to show the applicability of the obtained results.

Keywords. Oscillations; Averaging Techniques; Alternating Coefficients; Nonlinear Differential Equations; Second Order.

1. Introduction

This research is dealing with the oscillation property of the following second order nonlinear differential equation

$$\left(r(t)f(\dot{x}(t))\right)^{\bullet}+q(t)|x(t)|^{\gamma}\operatorname{sign}x(t)=0\ , \qquad \gamma>0 \tag{E}$$

where q and r are continuous functions on the interval $[t_0,\infty)$, $t_0\geq t$, $r(t)$ is a positive function on $\mathbb{R}$ and $\gamma>0$, f is a continuous function also on $\mathbb{R}$. Our attention here is concentrated only to such solution $x(t)$ of the differential equation (E) which exists on some interval $[t_0,\infty)$, $t_0\geq 0$.

In general, the theory of the oscillation of second order differential equations with alternating coefficients has been extensively investigated. Many papers motivated an especial technique depending on the averaging characteristic of the integrals of the alternating coefficients, see for instance the papers of Kamenev [8], Butler [4], Grace [6], Wong [13], and Yan [14] and the references cited therein. However, readers are invited to have an extensive look on the literature concerning this object, see [3, 5, 7, 9-12]. Recently, Ahmed and Dinar [2] and Ahmed et al. [1] studied the Eq. (E) in the case when $f(y)\equiv y$ and derived some oscillation criterions with respect to different ranges of γ, that is ($0<\gamma<1$ or $\gamma>1$).

In fact, we feel that it is of interest to establish new oscillation criterions using the known Kamenev's type condition with removing from γ some restrictions about its range of values. During this study, we proceed the discussion in a way similar to that in Yan [14, Theorem 2]. We also posed an open problem that related to the possible values of γ.

As we mentioned before, to simplify the proofs of our main results here, the main tools largely involved are integral averaging techniques with the generalized Riccati transformation.

As usual, the solution $x(t)$ of Eq. (E) is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is positive or negative eventually. In the sequence, Eq. (E) is called oscillatory if all its solutions are oscillatory.

2. Main Results

* Department of Mathematics, College of Education, University of Sirte, Libya

zahra80zahra@yahoo.com

Theorem 2.1: Suppose that

(H_1) $f(y) = yg(y);\ 0 \le A \le g(y) \le B$

(H_2) $\liminf\limits_{t\to\infty} \int_{t_0}^{t} q(s)\,ds > -\infty$,

(H_3) $\int^{\infty} \frac{ds}{r(s)} = \infty$,

Suppose in addition that there exists a continuous function

$$\phi : [t_0, \infty) \to \Re \text{ and } \alpha \in (1, \infty)$$

such that

(H_4) $\limsup\limits_{t\to\infty} \frac{1}{t^{\alpha}} \int_{t_0}^{t} (t-u)^{\alpha} q(u)\,du < \infty$,

and for every $\theta > 0$

(H_5) $\liminf\limits_{t\to\infty} \frac{1}{t^{\alpha}} \int_{s}^{t} (t-u)^{\alpha-2} \left[(t-u)^2 q(u) - \frac{\theta\alpha^2}{4\gamma a(u)} \right] du \ge \phi(s)$,

(H_6) $\int^{\infty} a(s)\phi_+^2(s)ds = \infty$,

where

$$\phi_+(t) = \max\{\phi(t), 0\} \text{ and } a(t) = \frac{1}{r(t)} \left(\int_{t_0}^{t} \frac{ds}{r(s)} \right)^{-1}.$$

Then equation (E) is oscillatory for $\gamma > 0\, and\, \gamma \ne 1$.

Proof: For the sake of a contradiction, we assume that there exists a solution $x(t)$ which may assumed to be eventually positive on the interval $[T_1, \infty)$ for some $T_1 \ge t_0 \ge 0$. [if the solution $x(t)$ is eventually negative, the proof is similar]. Define

$$w(t) = \frac{r(t) f(\dot{x}(t))}{x^{\gamma}(t)} \quad for\, t \ge T_1$$

Using Eq. (E) we get

$$\dot{w}(t) = -q(t) - \gamma r(t) \frac{\dot{x}(t) f(\dot{x}(t))}{x^{\gamma+1}(t)}, t \ge T_1 \tag{2.1}$$

which implies that

$$\frac{r(t)f(\dot{x}(t))}{x^{\gamma}(t)} + \gamma \int_{T_1}^{t} \frac{r(s)\dot{x}(s)f(\dot{x}(s))}{x^{\gamma+1}(s)} ds + \int_{T_1}^{t} q(s)ds = C_1, \quad C_1 = \frac{r(T_1)f(\dot{x}(T_1))}{x^{\gamma}(T_1)} \tag{2.2}$$

By using condition (H_1) we obtain that

$$\frac{r(t)\dot{x}(t)g(\dot{x}(t))}{x^{\gamma}(t)} + \int_{T_1}^{t} q(s)ds + \gamma \int_{T_1}^{t} \frac{r(s)\dot{x}^2(s)g(\dot{x}(s))}{x^{\gamma+1}(s)} ds = C_1, \quad t \ge T_1$$

Also by using (H_1), since $0 < A < g(y)$, then we have

$$\frac{r(t)\dot{x}(t)g(\dot{x}(t))}{x^{\gamma}(t)} + \int_{T_1}^{t} q(s)ds + \gamma A \int_{T_1}^{t} r(s) \left(\frac{\dot{x}(s)}{x^{\beta}(s)} \right)^2 ds \le C_1; \quad t \ge T_1, \ \beta = (\gamma+1)/2 \tag{2.3}$$

Or we can get from the equation (2.2) that

$$w(t)+\int_{T_1}^{t} q(s)ds+\gamma\int_{T_1}^{t}\frac{x^{2(\beta-1)}(s)}{r(s)g(\dot{x}(s))}w^2(s)ds=C_1, \quad t\geq T_1, \tag{2.4}$$

By employing the condition (H_1), since $g(y)\leq B$, Then (2.4) becomes

$$w(t)+\int_{T_1}^{t} q(s)ds+\frac{\gamma}{B}\int_{T_1}^{t}\frac{x^{2(\beta-1)}(s)}{r(s)g(\dot{x}(s))}w^2(s)ds\leq C_1, \quad t\geq T_1,$$

Now, for the behaviour of $\dot{x}(t)$, consider the following three cases:

Case1. $\dot{x}(t)$ is oscillatory. Then there exists a sequence $\{t_m\}_{m=1}^{\infty}$ in $[T_1,\infty)$ with $\lim\limits_{m\to\infty} t_m=\infty$ such that $\dot{x}(t_m)=0$, ($m=1,2,3,\ldots$). Thus (2.3) gives

$$\gamma A\int_{T_1}^{t_m} r(s)\left(\frac{\dot{x}(s)}{x^{\beta}(s)}\right)^2 ds\leq C_1-\int_{T_1}^{t_m} q(s)ds, \quad m=1,2,\ldots.$$

By condition (H_2) we conclude that

$$\int_{T_1}^{\infty} r(s)\left(\frac{\dot{x}(s)}{x^{\beta}(s)}\right)^2 ds<\infty. \tag{2.5}$$

Therefore, for some constant $N>0$, we have

$$\int_{T_1}^{t} r(s)\left(\frac{\dot{x}(s)}{x^{\beta}(s)}\right)^2 ds\leq N \quad for\, t\geq T_1.$$

By using Schwarz inequality, we note that

$$\left|\int_{T_1}^{t}\left(\frac{\dot{x}(s)}{x^{\beta}(s)}\right)ds\right|^2=\left|\int_{T_1}^{t}\frac{1}{\sqrt{r(s)}}\sqrt{r(s)}\left(\frac{\dot{x}(s)}{x^{\beta}(s)}\right)ds\right|^2\leq\left(\int_{T_1}^{t}\frac{ds}{r(s)}\right)\left(\int_{T_1}^{t} r(s)\left(\frac{\dot{x}(s)}{x^{\beta}(s)}\right)^2 ds\right)\leq N\int_{T_1}^{t}\frac{ds}{r(s)} \tag{2.6}$$

On the other hand we have that

$$\left|\int_{T_1}^{t}\frac{\dot{x}(s)}{x^{\beta}(s)}ds\right|=\frac{1}{|1-\beta|}\left|x^{1-\beta}(t)-x^{1-\beta}(T_1)\right|,$$

which implies by (2.6) that

$$\left|x^{1-\beta}(t)-x^{1-\beta}(T_1)\right|\leq|1-\beta|\left(N\int_{T_1}^{t}\frac{ds}{r(s)}\right)^{\frac{1}{2}} \tag{2.7}$$

There exists $T_2\geq T_1$ and a positive constant M such that

$$\left|x^{1-\beta}(t)\right|\leq M\left(\int_{T_1}^{t}\frac{ds}{r(s)}\right)^{\frac{1}{2}} \quad for\, all\, t\geq T_2 \tag{2.8}$$

Using (2.8) in (2.1) we get

$$\dot{w}(t)\leq -q(t)-\frac{\gamma}{BM^2}a(t)w^2(t), \quad t\geq T_2 \tag{2.9}$$

where

$$a(t)=\frac{1}{r(t)}\left(\int_{T_1}^{t}\frac{ds}{r(s)}\right)^{-1},$$

and consequently, for all $t>s\geq T_2$, we obtain

$$\int_s^t (t-u)^{\alpha}\dot{w}(u)du \leq -\int_s^t (t-u)^{\alpha}q(u)du-\frac{\gamma}{BM^2}\int_s^t (t-u)^{\alpha}a(u)\,w^2(u)du$$

Now, since

$$\int_s^t (t-u)^{\alpha}\,\dot{w}(u)du=-(t-s)^{\alpha}w(s)+\alpha\int_s^t (t-u)^{\alpha-1}w(u)du$$

Then

$$\int_s^t (t-u)^{\alpha}q(u)du$$
$$\leq (t-s)^{\alpha}w(s)-\alpha\int_s^t (t-u)^{\alpha-1}\dot{w}(u)du-\frac{\gamma}{BM^2}\int_s^t (t-u)^{\alpha}a(u)\,w^2(u)du \tag{2.10}$$

So, we obtain

$$\int_s^t (t-u)^{\alpha}q(u)du$$
$$\leq (t-s)^{\alpha}w(s)-\int_s^t \left[\frac{\gamma}{BM^2}(t-u)^{\alpha}a(u)w^2(u)+\alpha(t-u)^{\alpha-1}w(u)\right]du \tag{2.11}$$

From which it follows that

$$\int_s^t (t-u)^{\alpha}q(u)-\frac{B\alpha^2M^2(t-u)^{\alpha-2}}{4\gamma a(u)}du$$
$$\leq (t-s)^{\alpha}w(s)$$
$$-\int_s^t\left[\frac{\gamma}{\sqrt{B}M}\ (t-u)^{\frac{\alpha}{2}}\sqrt{\gamma a(u)}w(u)+\frac{\alpha(t-u)^{\frac{\alpha}{2}-1}}{2\sqrt{\gamma a(u)}}w(u)\right]^2du$$
$$\leq (t-s)^{\alpha}w(s)\quad ,s\geq T_2 \tag{2.12}$$

Dividing (2.12) by t^{α}, we obtain that

$$\frac{1}{t^{\alpha}}\int_s^t (t-u)^{\alpha-2}\left[(t-u)^2q(u)-\frac{B\alpha^2M^2}{4\gamma a(u)}\right]du\ \leq\frac{(t-s)^{\alpha}}{t^{\alpha}}w(s) \tag{2.13}$$

Since

$$t^{-\alpha}\leq (t-s)^{-\alpha},\alpha\in(1,\infty)$$

Then

$$(t-s)^{\alpha}t^{-\alpha}\leq (t-s)^{\alpha}(t-s)^{-\alpha}=1$$

Substituting in (2.13) we obtain

$$\frac{1}{t^{\alpha}}\int_s^t (t-u)^{\alpha-2}\left[(t-u)^2q(u)-\frac{B\alpha^2M^2}{4\gamma a(u)}\right]du\leq w(s)$$

Clearly that by using (H_5), by taking the lower limit of above inequality as $t\to\infty$, we get

$$\phi(s) \le w(s)\ , s \ge T_2 ,$$

which implies that

$$\phi_+^2(s) \le w^2(s) \quad ; \qquad \phi_+(t) = \max\{\phi(t),0\}$$

Next, for all $t \geq T_2$, define the functions $y(t)$ and $z(t)$ as follows:

$$y(t) = t^{-\alpha} \int_{T_2}^{t} \alpha (t-u)^{\alpha-1} w(u) du ,$$

$$z(t) = t^{-\alpha} \int_{T_2}^{t} \frac{\gamma}{BM^2} (t-u)^{\alpha} a(u) w^2(u) du$$

Form (2.11), we get

$$y(t) + z(t) = t^{-\alpha} \int_{T_2}^{t} \left[\alpha (t-u)^{\alpha-1} w(u) + \frac{\gamma}{BM^2} (t-u)^{\alpha} a(u) w^2(u) \right] du$$

$$\le t^{-\alpha} (t-T_2)^{\alpha} w(T_2) - t^{-\alpha} \int_{T_2}^{t} (t-u)^{\alpha} q(u) du$$

(2.14)

and we see that (H_5) implies that

$$\liminf_{t\to\infty} t^{-\alpha} \int_{s}^{t} (t-u)^{\alpha} q(u) du \ge \phi(s) , \qquad s \ge T_2 \tag{2.15}$$

Since

$$\limsup_{t\to\infty} t^{-\alpha} \int_{T_2}^{t} (t-u)^{\alpha} q(u) du \ge \liminf_{t\to\infty} t^{-\alpha} \int_{T_2}^{t} (t-u)^{\alpha} q(u) du$$

Hence, by taking into account (H_5), we conclude

$$\lim_{t\to\infty} sup\ t^{-\alpha} \int_{T_2}^{t} (t-u)^{\alpha} q(u) du \ge \emptyset(T_2) + \lim_{t\to\infty} inf \frac{BM^2}{4\gamma} t^{-\alpha} \int_{T_2}^{t} \frac{\alpha^2}{a(u)} (t-u)^{\alpha-2} du$$

(2.16)

Together with (H_4); (2.16) shows that there exists a sequence

$$\{\tau_n\}_{n=1,..}, \qquad \tau_n > T_2, n = 1,2,........., \qquad \lim_{t\to\infty} \tau_n = \infty$$

such that

$$\lim_{n\to\infty} \frac{BM^2}{4\gamma} t^{-\alpha} \int_{T_2}^{\tau_n} \frac{\alpha^2}{a(u)} (t-u)^{\alpha-2} du < \infty \tag{2.17}$$

Next, by taking the upper limit as $t \to \infty$ in (2.14) and using (2.17), we get

$$\limsup_{t\to\infty} [y(t) + z(t)] \le w(T_2) - \limsup_{t\to\infty} t^{-\alpha} \int_{T_2}^{t} (t-u)^{\alpha} q(u) du$$

$$\le w(T_2) - \liminf_{t\to\infty} t^{-\alpha} \int_{T_2}^{t} (t-u)^{\alpha} q(u) du$$

$$\le w(T_2) - \phi(T_2) = k$$

Hence for all sufficiently large n,

$$y(\tau_n) + z(\tau_n) < k \tag{2.18}$$

Since

$$z(t)=\frac{\gamma}{M^2}\int_{T_2}^{t}\left(1-\frac{u}{t}\right)^{\alpha}a(u)w^2(u)du>0$$

Then by the Leibnitz rule, we can see that

$$\dot{z}(t)=\frac{\alpha\gamma}{BM^2}\int_{T_2}^{t}\left(1-\frac{u}{t}\right)^{\alpha-1}\frac{u}{t^2}a(u)w^2(u)du>0$$

Since $z(t)$ is a positive increasing function on $t\geq T_2$, we see that $\lim_{t\to\infty}z(t)=b$, where $b=\infty$ or is a positive constant. Suppose that $b=\infty$, then $\lim_{n\to\infty}z(\tau_n)=\infty$ and by using (2.18) it follows that

$$\lim_{n\to\infty}y(\tau_n)<k-\lim_{n\to\infty}z(\tau_n)<k-\infty=-\infty \tag{2.19}$$

By (2.18) and (2.19) it can be shown that, for all large τ_n,

$$\left[\frac{y(\tau_n)}{z(\tau_n)}\right]<\sigma-1<0,\ 0<\sigma<1, z(\tau_n)>0 \tag{2.20}$$

On the other hand, by Schwarz inequality, we have for all large τ_n that

$$\begin{aligned}0\leq y^2(\tau_n)&=\tau_n^{-2\alpha}\left(\int_{T_2}^{\tau_n}\alpha(\tau_n-u)^{\alpha-1}\sqrt{\frac{a(u)}{a(u)}}w(u)du\right)^2\\&\leq\left(\tau_n^{-\alpha}\frac{BM^2}{\gamma}\int_{T_2}^{\tau_n}\alpha^2(\tau_n-u)^{\alpha-2}\frac{1}{a(u)}du\right)\\&\times\left(\tau_n^{-\alpha}\frac{\gamma}{BM^2}\int_{T_2}^{\tau_n}\alpha^2(\tau_n-u)^{\alpha}a(u)w^2(u)du\right)\end{aligned}$$

Therefore,

$$0\leq\frac{y^2(\tau_n)}{z(\tau_n)}\leq\frac{BM^2}{\gamma}\tau_n^{-\alpha}\int_{T_2}^{\tau_n}\frac{\alpha^2}{a(u)}(\tau_n-u)^{\alpha-2}du$$

This implies that

$$0\leq\lim_{n\to\infty}\frac{(y(\tau_n))^2}{z(\tau_n)}=\lim_{n\to\infty}y(\tau_n)\times\lim_{n\to\infty}\frac{y(\tau_n)}{z(\tau_n)}\leq\lim_{n\to\infty}\frac{BM^2}{\gamma}\tau_n^{-\alpha}\int_{T_2}^{\tau_n}\frac{\alpha^2}{a(u)}(\tau_n-u)^{\alpha-2}du \tag{2.21}$$

But we can get by (2.17) that

$$\frac{BM^2}{\gamma}\lim_{n\to\infty}\tau_n^{-\alpha}\int_{T_2}^{\tau_n}\frac{\alpha^2}{a(u)}(\tau_n-u)^{\alpha-2}du<\infty$$

Substituting in (2.21), it follows

$$0\leq\lim_{n\to\infty}y(\tau_n)\times\lim_{n\to\infty}\frac{y(\tau_n)}{z(\tau_n)}<\infty.$$

But this is a contradiction with both of (2.19) and (2.20) which gives us the fact that

$$\lim_{t\to\infty}z(t)=b<\infty.$$

By taking into consideration the inequality (2.13), we have

$\lim_{t\to\infty} t^{-\alpha}\int_{T_2}^{t}(t-u)^{\alpha}a(u)\emptyset_+{}^2(u)du \le \lim_{t\to\infty} t^{-\alpha}\int_{T_2}^{t}(t-u)^{\alpha}\, a(u)w^2(u)du\ =$
$\frac{BM^2}{\gamma}\lim_{t\to\infty} z(t) < \infty$

which contradicts the condition (H_6).

Case 2. $\dot{x}(t) > 0$ on $[T_3,\infty)$ for some $T_3 \ge T_1$. Thus $\frac{r(t)\dot{x}(t)g(\dot{x}(t))}{x^{\gamma}(t)}$ is positive. By the condition (H_2) and (2.3) it follows that (2.5) holds, and hence we can complete the proof by following the same procedure of case1.

Case 3. $\dot{x}(t) < 0$ on $[T_3,\infty)$ for some $T_3 \ge T_1$. If (2.5) holds, then we can arrive at a contradiction by the procedure of case1. So, we suppose that the integral in (2.5) diverges. Using (H_2) in (2.3) we have the following: Since

$$-\limsup_{t\to\infty}\int_{T_1}^{t} q(s)ds < -\liminf_{t\to\infty}\int_{T_1}^{t} q(s)ds < \infty$$

Then there exists a constant L such that

$$\int_{T_1}^{t} q(s)ds \ge -L$$

Since $g(y) \le B$ and $\dot{x}(t) < 0$, then we obtain that

$$\frac{-r(t)\dot{x}(t)}{x^{\gamma}(t)} \ge -C + \frac{\gamma A}{B}\int_{T_1}^{t}\frac{r(s)\dot{x}^2(s)}{x^{\gamma+1}(s)}ds, \quad C = \frac{C_1+L}{B}$$

Now, choose $T \ge T_3$ such that

$$\frac{\gamma A}{B}\int_{T_3}^{t}\frac{r(s)\dot{x}^2(s)}{x^{\gamma+1}(s)}ds = 1 + C$$

which implies that

$$\frac{-r(t)\dot{x}(t)\big/x^{\gamma}(t)}{-C+\frac{\gamma A}{B}\int_{T_3}^{t}\frac{r(s)\dot{x}^2(s)}{x^{\gamma+1}(s)}ds}\left(-\frac{\gamma A\dot{x}(t)}{Bx(t)}\right) \ge -\frac{\gamma A}{B}\frac{\dot{x}(t)}{x(t)}$$

Integrating the above inequality from T to t we obtain that

$$\int_{T}^{t}\frac{\gamma A r(s)\dot{x}^2(s)\big/Bx^{\gamma+1}(s)}{-C+\frac{\gamma A}{B}\int_{T_3}^{t}\frac{r(u)\dot{x}^2(u)}{x^{\gamma+1}(u)}du}ds \ge -\frac{\gamma A}{B}\int_{T}^{t}\frac{\dot{x}(s)}{x(s)}\,ds$$

That is

$$\ln\left[-C+\frac{\gamma A}{B}\int_{T_3}^{t}\frac{r(u)\dot{x}^2(u)}{x^{\gamma+1}(u)}du\right] \ge \ln\left(\frac{x(T)}{x(t)}\right)^{\frac{\gamma A}{B}},$$

which together with (2.18) yields

$$\frac{-r(t)\dot{x}(t)}{x^{\gamma}(t)} \geq \left(\frac{x(T)}{x(t)}\right)^{\frac{\gamma A}{B}} \tag{2.22}$$

Now, since $\dot{x}(t) < 0$ for $t \geq T_3 \geq T_1$, then, we have

$$x(t) < x(T)\ ,T \geq T_3 \quad \text{and} \quad x^{\frac{\gamma A}{B}}(t) < x^{\frac{\gamma A}{B}}(T) \tag{2.23}$$

But we assumed that $x(t) > 0$, then $x^{\gamma\frac{A_2}{A_3}}(t) > 0$. Thus, (2.23) gives us that

$$\left(\frac{x(T)}{x(t)}\right)^{\frac{\gamma A}{B}} > 1$$

Substituting in inequality (2.22) we obtain that

$$-r(t)\frac{\dot{x}(t)}{x^{\gamma}(t)} > 1$$

So we have

$$-r(t)\dot{x}(t) > x^{\gamma}(t) \tag{2.24}$$

Since $x(t) > 0\ ,t \geq T_1$ .Then for all $t \geq T_1$ there exists $\theta > 0$ such that $x(t) > \theta$. In (2.24) we get

$$-r(t)\dot{x}(t) > \theta^{\gamma}\ ,$$

which it follows that

$$x(t) < x(T) - \theta^{\gamma}\int_{T}^{t}\frac{ds}{r(s)} \tag{2.25}$$

Taking the limit of (2.25) as $t \to \infty$ we obtain by using the condition (H_3) that $\lim_{t\to\infty} x(t) = -\infty$,but this contradicts the fact that $x(t) > 0$ for $t \geq T_1$ and the proof is complete.

Example 2.1 Consider the differential equation

$$\left(\frac{1}{t}\left(\dot{x}(t) + \frac{\dot{x}(t)e^{\dot{x}(t)}}{e^{\dot{x}(t)}+1}\right)\right)^{\bullet} + \cos t\,|x(t)|^{\gamma}\, sign x(t) = 0, \qquad t \geq t_0 > 0\ ,$$

$$\gamma > 0 \ \ and \ \ \gamma \neq 1\,.$$

We note that

(i) *Since*

$$f(\dot{x}(t)) = \dot{x}(t) + \frac{\dot{x}(t)e^{\dot{x}(t)}}{e^{\dot{x}(t)}+1}.$$

Then

$$1 < g(\dot{x}(t)) = 1 + \frac{e^{\dot{x}(t)}}{e^{\dot{x}(t)}+1} < 2 \quad \text{for all} \quad \dot{x}(t) \in \Re,$$

(ii) $\liminf\limits_{t\to\infty} \int_{t_0}^{t} q(s)\,ds = \liminf\limits_{t\to\infty} \int_{t_0}^{t} \cos s\,ds$

$$= \liminf_{t\to\infty} \left[\sin t - \sin t_0\right]$$

$$> -\infty ,$$

(iii) $\int^{\infty} \frac{1}{r(s)}\,ds = \int^{\infty} s\,ds = \infty$,

taking $\alpha = 2$, we have

$(iiii)$ $\limsup\limits_{t\to\infty} \frac{1}{t^2}\int_{t_0}^{t} (t-u)^2 q(u)\,du = \limsup\limits_{t\to\infty} \frac{1}{t^2}\int_{t_0}^{t} (t-u)^2 \cos u\,du$

$$= \limsup_{t\to\infty} \frac{1}{t^2}\Big[t^2 \sin u - 2tu\sin u - 2t\cos u + u^2 \sin u$$

$$+ 2u\cos u - 2\sin u\,\Big]\Big|_{t_0}^{t} = -\sin t_0 < \infty,$$

(iv) $\liminf\limits_{t\to\infty} \frac{1}{t^2}\int_{s}^{t} \left[(t-u)^2 \cos u - \frac{\theta}{\gamma u}\right] du \geq -\sin s - k,$

where k is a positive constant . Set $\phi(s) = -\sin s - k$, and choose an integer N such that

$$\left(2N+1\right)\pi + \frac{\pi}{4} \geq t_0 \ .$$

Then for all integers $n \geq N$ and $\left(2n+1\right)\pi + \frac{\pi}{4} \leq s \leq 2(n+1)\pi - \frac{\pi}{4},$ we have

$$\phi(s) = -\sin s - k \geq \delta s \ ,$$

where δ is a small constant . Thus,

$$\lim_{t\to\infty}\int_{t_0}^{t} a(s)\phi_+{}^2(s)\,ds \geq \sum_{n=N}^{\infty} \delta^2 \int_{(2n+1)\pi+\pi/4}^{2(n+1)\pi-\pi/4} s\,ds = \infty \ .$$

Hence by Theorem 2.1, we conclude that the given equation is oscillatory.

The following result is concerned with the oscillatory solution of equation (E) when the hypothesis (H_4) fails.

Theorem 2.2 Suppose that there exists a constant $\alpha \in (1,\infty)$ such that hypothesis (H_1)-(H_3) hold and for every $\theta > 0$

(H_7) $$\limsup_{t\to\infty} \frac{1}{t^{\alpha}} \int_{t_0}^{t} (t-u)^{\alpha-2}\left[(t-u)^2 q(u) - \frac{\theta\alpha^2}{4\gamma a(u)}\right] du = \infty ,$$

where

$$a(t) = \frac{1}{r(t)}\left(\int_{t_0}^{t} \frac{ds}{r(s)}\right)^{-1}$$

Then Eq. (E) is oscillatory for $\gamma > 0$ *and* $\gamma \neq 1$.

Proof: For the sake of a contradiction, we assume that there exists a solution $x(t)$ which may assumed to be eventually positive on the interval $[T_1,\infty)$ for some $T_1 \geq t_0 \geq 0$. [if the solution $x(t)$ is eventually negative, the proof is similar]. As in the proof of Theorem 2.1 (case1), we obtain (2.12). Dividing it by t^{α} and take the upper limit as $t \to \infty$, we obtain

$$\lim_{t\to\infty} sup\, t^{-\alpha} \int_s^t (t-u)^{\alpha-2}\left((t-u)^2 q(u) - \frac{B\alpha^2 M^2}{4\gamma a(u)}\right) du \leq w(s)$$

Letting $s = T_2$ in the above inequality to get

$$\lim_{t\to\infty} sup\, t^{-\alpha} \int_{T_2}^t (t-u)^{\alpha-2}\left((t-u)^2 q(u) - \frac{B\alpha^2 M^2}{4\gamma a(u)}\right) du \leq w(T_2)$$

which leads to a contradiction for the condition (H_7).

The proofs in the cases when $\dot{x}$ is either positive or negative on $[T_2,\infty)$ $, T_2 \geq T_1$ are similar to the proofs in cases 2 and 3 of Theorem 2.1, thus they will be omitted.

Corollary 2.1: Let condition (H_7) in Theorem 2.2 be replaced by

(H_8) $$\limsup_{t\to\infty} \frac{1}{t^{\alpha}} \int_{t_0}^{t} (t-u)^{\alpha} q(u)\, du = \infty ,$$

(H_9) $$\limsup_{t\to\infty} \frac{1}{t^{\alpha}} \int_{t}^{t} \frac{(t-u)^{\alpha-2}}{a(u)} \alpha^2 du < \infty .$$

Then the conclusion of Theorem 2.2 holds.

Proof: : For the sake of a contradiction, we assume that there exists a solution $x(t)$ a) which may assumed to be eventually positive on the interval $[T_1,\infty)$ for some $T_1 \geq t_0 \geq 0$. [if the solution $x(t)$ is eventually negative, the proof is similar]. As in the proof of Theorem 2.1 (case1), we obtain (2.12), and by dividing it by t^{α} and take the upper limit as $t \to \infty$, we obtain

$$\lim_{t\to\infty} sup\, t^{-\alpha} \int_s^t (t-u)^{\alpha} q(u) du \leq \lim_{t\to\infty} \sup \frac{BM^2}{4\gamma} t^{-\alpha} \int_s^t \frac{\alpha^2 (t-u)^{\alpha-2}}{a(u)} du + w(s)$$

Or

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha}} \int_{T_2}^{t} (t-u)^{\alpha} q(u) du \leq \lim_{t\to\infty} \sup \frac{BM^2}{4\gamma} t^{-\alpha} \int_s^t \frac{\alpha^2 (t-u)^{\alpha-2}}{a(u)} du + w(T_2)$$

which with condition (H_9) implies that

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha}} \int_{T_2}^{t} (t-u)^{\alpha} q(u)\, du < \infty ,$$

But this is a contradiction with the condition (H_8). The proofs of cases 2 and 3 are immediate consequences of cases 2 and 3 of Theorem 2.1 and hence they will be omitted.

Remark 2.1: Theorems 2.1 and 2.2 extend and improve some of previous results in the literature. See Grace [6], Wong [12, 13], Philos [10], Yan [14] and the recent papers of Ahmed and Dinar [2] and Ahmed et al. [1].

3. Conclusion

In conclusion, by the generalized Riccati transformation and the integral averages techniques, some new sufficient conditions are derived. The results obtained here are valid for the oscillation of Eq. (E) for all $\gamma > 0$ *and* $\gamma \neq 1$. So, we think that, it will be of an interest to study the oscillation of Eq. (E) in the case when $\gamma = 1$ and give sufficient conditions for the oscillation of all solutions.

4. Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

5. Acknowledgment

Authors of this manuscript would like to thank the reviewers for their useful comments and suggestions which help to improve the content of their scholar.

6. References

[1] Ahmed, F. N., Abumaryam S. A., Saad, A. I., (2022), Oscillation criterion for some differential equations of second order with integrable coefficients, The 1st International conference of The Faculties of Sciences, University of Zawia, Libya.

[2] Ahmed, F. N. and Ali, D. A., (2019), On the oscillation property for some nonlinear differential equations of second order, Journal of Pure & Applied Sciences, Sebha University, Libya, Vol. 18,342-345.

[3] Bartle, R.G. The elements of real analysis, 7th ed., John Wiley and Sons, 233, 1976.

[4] Butler, G. J., (1980), Integral averages and the oscillation of second order ordinary differential equations, SIAM J. Math. Anal. **11**, 190–200.

[5] Elabbasy, E. M., and Elzeiny; Sh. R., (2011), Oscillation theorems concerning non-linear differential equations of the second order, Opuscula Mathematica, **31,** 373-391.

[6] Grace, S.R., (1990), Oscillation criteria for second order differential equations with damping, J. Austral. Math. Soc. **49**, 43–54.

[7] Grace, S. R. and Lalli; B. S., (1990), Integral averaging technique for the oscillation of second order nonlinear differential equations. J. Math. Anal. Appl. **149**, 277-311.

[8] Kamenev, I.V., (1978), Integral criterion for oscillation of linear differential equations of second order, Math. Zametki, **23,** 249–251.

[9] Li; H. J., (1995), Oscillation criteria for second order linear differential equations, J. Math. Anal. Appl. 194, 217-234.

[10] Philos, Ch. G., (1986) 'Integral averaging techniques for the oscillation of second order sublinear ordinary differential equations', J. Austral. Math. Soc. Ser. A 40, 111-130.

[11] Philos; Ch. G., (1989), Oscillation theorems for linear differential equations of second order, Arch. Math. **53,** 483-492.

[12] Wong, J.S.W., (1969), Oscillation and non-oscillation of solutions of second order linear differential equations with integrable coefficients. Trans. Amer. Math. Soc., 144, 197-215

[13] Wong, J. S. W., (1986), An oscillation criterion for second order nonlinear differential equations, Proc. Amer. Math. Soc. 98, 109–112.

[14] Yan J., (1986), Oscillation theorems for second order linear differential equations with damping, Proc. Amer. Math. Soc. **98**, 267-282.