The Conformal Mapping between the Globally as well as Locally of the Earth (Ellipsoid) and the Cartesian Coordinates

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Abstract

We investigate the geometrical properties of the ellipsoid (by ellipsoid we mean an ellipsoid of revolution) globally as well as locally. on this curved surface we study the basic transformation between polar and curvilinear coordinates . the series expansion contain a sufficient number of terms to guarantee computational accuracies at maximum level .we start by deriving the length of the meridian arc form equator to a point with arbitrary latitude. The computation as fundamental to all conformal mapping. we will derive the explicit formulas for the transformation: (latitude ,longitude) $= (\phi,\lambda) \rightarrow (x, y)$ and $(x, y) \rightarrow (\phi, \lambda)$ .These formulas are adopted for the Universal Transverse Mercator Mapping (UTM) . As an application of this research work we give examples for each case of work , and we take three locations in the Misurata City which are the Misurata University , the Faculty of Science , and the Stadium in Misurata City.

Keywords: Conformal mapping, Universal Transverse Mercator Mapping (UTM).

1. Introduction

In the early18th century, Isaac Newton and others concluded that the Earth should be slightly flattened at the poles , but the French Academy of Sciences ,(1735) believed the Earth to be egged-shaped as the result of measurements within France. This established the validity of Newton's conclusions and led to numerous meridian measurements in various locations , especially during the 19th and 20th centuries ,between1799 and 1951there were 26 determinations of dimensions of the Earth .All of the listed for values for the size and shape of the Earth are different .An ellipsoid is

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defined as the solid(i.e. three dimensional object) produced by rotating an ellipse (a two dimensional object) about one of its axes .The ellipsoid is usually an oblate (flattened) spheroid with two different axes ,an equatorial radius (the semi- major axis a)and a polar radius (the semi-minor axis b) . In the case of the Earth , the ellipse is rotated about the minor axes .An oblate ellipsoid is defined as an ellipsoid which is flattened at the poles .Thus ,the term oblate applies only to an object which is rotating , as the Earth does ,since the term pole would have no meaning otherwise . (The term prolate ellipsoid would apply to an ellipsoid which is expanded at the poles).Thus ,all of these measurements pretty much confirmed that the shape of the Earth is that of an oblate ellipsoid .That the Earth is an oblate , rather than prolate .The most of the Reference Ellipsoid is given by a , b and f (flattened). Table 1. quotes values for frequently used ellipsoids.

Table 1. Parameters for reference ellipsoids

Name	a $[m]$	b $[m]$	$f=(a-b)/a$
International 1924	6378388	6356911.946	1/297
Krassovsky 1948	6378245	6356863.019	1/298.3
International 1980	6378137	6356752.298	1/298.257
World Geodetic System 1984	6378137	6356752.314	1/298.257223563

In the case of the Ellipsoid we are concerned with here , the Earth , the major ,or semi-major , radius is defined by the radius of the equator , the equatorial radius. The minor axis, or semi-minor axis, is defined by the distance from the center of the Earth to either pole, and is referred to as the polar radius Being an oblate ellipsoid , the polar radius is required to be somewhat less than the equatorial radius. It is possible for many different coordinate systems to be defined a primary use of ellipsoids is to serve as a basis for a coordinate system of latitude (north/south) ,longitude (east /west) and elevation (height)for this purpose it is necessary to identify a zero meridian ,which for earth is usually the prime meridian (Green-wich).The longitude measures the rotational angle between the zero meridian and the measured point ,by convention for the earth it is expressed as degrees ranging from -180 to $+180$ The latitude measures how close to the poles or equator appoint is along a meridian. And is represented as angle from -90 to $+90$ .Where 0^0 is the equator. Since the Earth is an ellipsoid of revolution , and the shape of the Earth is determined by shape the ellipse which is rotated .

2. The Ellipsoid of Revolution

The equation of the ellipse in x , y coordinate system is , see Figure 1

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \tag{2.1}$$

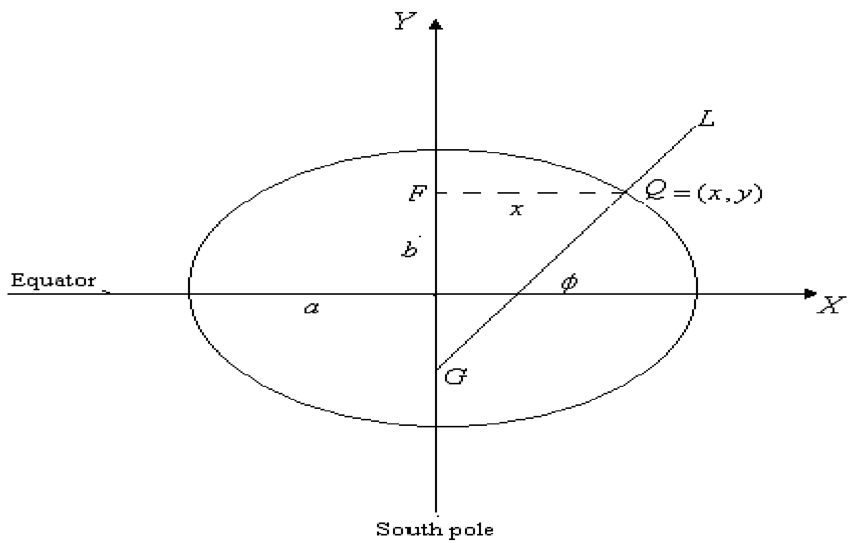

Figure 1. The Ellipse

This meridian ellipse is determined by it is semi major a and semi minor b axes . However , more often the dimensionless quantities flattening f , first eccentricity e and second eccentricity e' are used .We give a complete list of the ties among all these quantities :

$$\left.\begin{aligned}
&\frac{a-b}{a} = f = 1-\sqrt{1-e^2} = 1-\frac{1}{\sqrt{1+e'^2}} \\
&\frac{a^2-b^2}{a^2} = f(2-f) = e^2 = \frac{e'^2}{1+e'^2} \\
&\frac{a-b}{a+b} = \frac{f}{2-f} = \frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}} = \frac{\sqrt{1+e'^2}-1}{\sqrt{1+e'^2}+1} \\
&\frac{a^2-b^2}{a^2+b^2} = \frac{f(2-f)}{1+(1-f)^2} = \frac{e^2}{2-e^2} = \frac{e'^2}{2+e'^2} \\
&\frac{b}{a} = \frac{a}{b} = 1-f = \sqrt{1-e^2} = \frac{1}{\sqrt{1+e'^2}}
\end{aligned}\right\} \tag{2.2}$$

We also use the radius of curvature $c = a^2/b$ at the poles :

$$c = \frac{a^2}{b} = a\sqrt{1+e'^2} = b(1+e'^2) \tag{2.3}$$

A point q on the ellipsoid is determined by (latitude ,longitude) = (ϕ , λ). The geographical latitude ϕ is the angle between the normal at q and the plane of the Equator . For an ellipse , the normal at q does not contain the origin. (This is the price we must pay for flattening : ϕ is not the angle from the center of the ellipse) . The geographical longitude λ is the angle between the plane of the meridian of q ,See Figure 2.

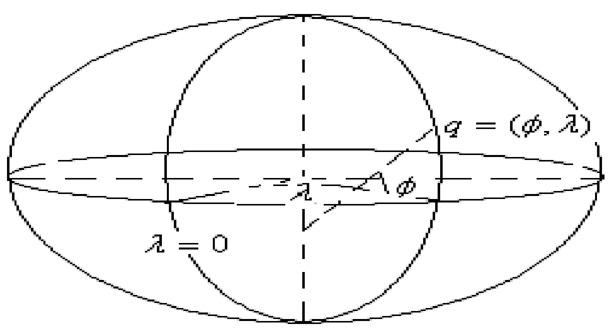

Figure 2. The parameter system of the ellipsoid

3. Principal Curvatures

The Curvature is the state and degree of deviation from straight line i.e. an (arced line).A plane that does not contain the surface normal is inclined . The radius of curvature for an inclined plane equals the radius of Curvature in the corresponding normal section R times cosine of angle between the planes . This is called Meusnier's Theorem .

$$\rho = R\cos\theta \tag{3.1}$$

To determine the principal curvature for the normal sections of the ellipsoid we only need a few simple geometrical considerations .Any meridian plane is a plane of symmetry, so the direction of the meridian is a direction of principal curvature. We call the radius of curvature M in this direction and the radius of curvature in the orthogonal direction is called N .The ellipsoid is flattened at the poles and accordingly the curvature at any point is larger in the direction of the meridian so M is smaller than N . N is also called the prime vertical radius of curvature . From Figure 3 we have $dB = Md\phi$ or $M = dB/d\phi$. We want to express the length of arc dB as a function of latitude ϕ and start from equation (2.1) for the ellipse . The relation between latitude ϕ and the Cartesian coordinates (x , y) of a point of the ellipse is given as :

$$\tan\phi = -\frac{dx}{dy} \tag{3.2}$$

From (2.1) we may express x explicitly as a function of y .We differentiate and insert into (3.2) then :

$$\tan\phi = \left(\frac{a}{b}\right)^2 \frac{y}{x} \tag{3.3}$$

We want to eliminate y and insert from (2.1) and use (2.2) and get :

$$x = \frac{a\sqrt{1+e'^2}\cos\phi}{\sqrt{1+e'^2\cos^2\phi}} \tag{3.4}$$

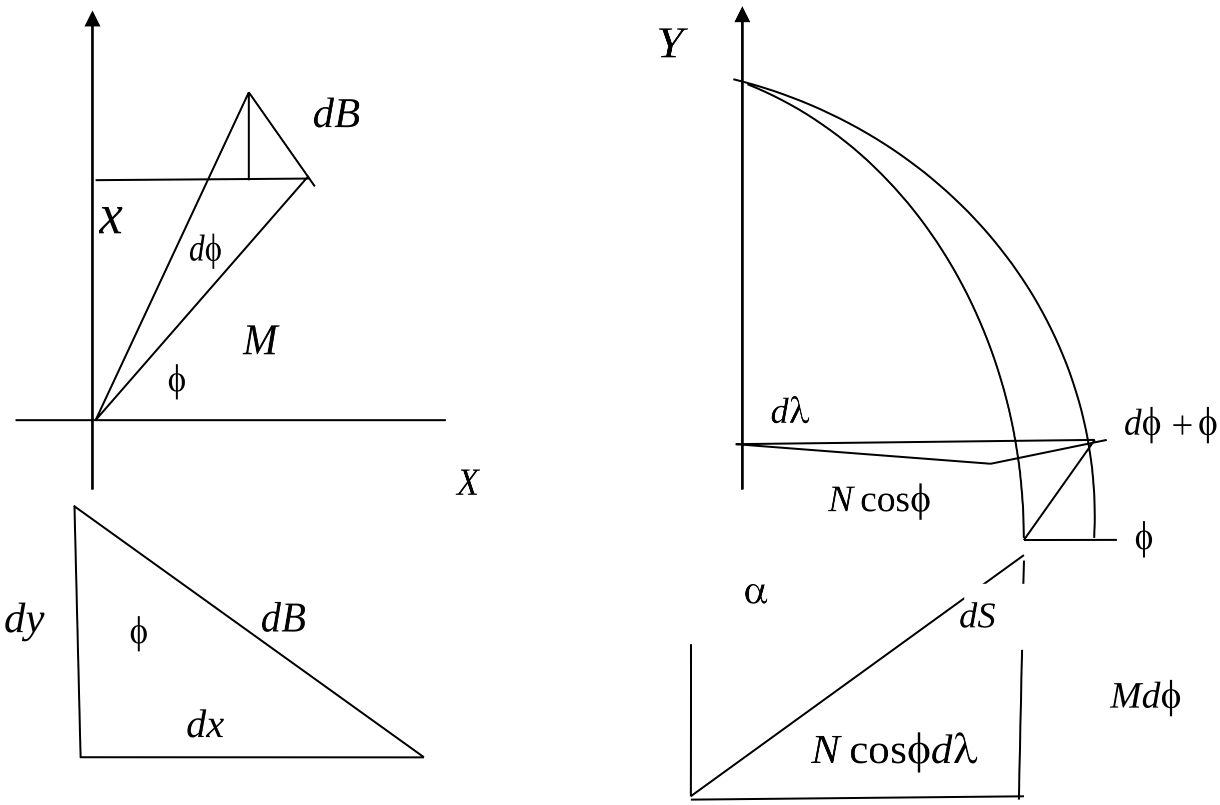

Figure 3. left :The differential triangle for the meridian with an increase $d\phi$ in latitude . Right: The differential triangle belonging to an increase $d\lambda$ in longitude .Below :The shaded figures are shown in an enlarged scale

According to Figure 3.1 we have $dB = -dx/\sin\phi$ and get by insertion into (3.4) then :

$$M = \frac{dB}{d\phi} = \frac{dB}{dx}\frac{dx}{d\phi} = \frac{-1}{\sin\phi}\frac{dx}{d\phi} = \frac{-1}{\sin\phi}\Big(\sqrt{1+e'^2\cos^2\phi}\,(-a\sin\phi\,\sqrt{1+e'^2}\,)$$

$$+a\sqrt{1+e'^2}\left(1+e'^2\cos^2\phi\right)e'^2\cos^2\phi\sin\phi\Big)\Big/\left(1+e'^2\cos^2\phi\right)$$

$$= \frac{c}{\left(1+e'^2\cos^2\phi\right)^{3/2}} \qquad (3.5)$$

Auxiliary Variables : We start by defining the variables V and η :

$v = \sqrt{1+\eta^2}, \eta = e'\cos\phi$

And then :

$$v^2 = 1 + \eta^2 = 1 + e'^2 \cos^2 \phi \qquad (3.6)$$

Now (3.5) can be expressed as :

$$M = \frac{c}{V^3} \qquad (3.7)$$

In order to calculate N we consider at point Q both the circle of the parallel and the circle of curvature .They lie in different planes the traces of which in Figure 2. are the lines QG and QF .The line $QF = x$ section is radius for the parallel and N is the radius of curvature in the normal section at Q, they make the angle ϕ and according to the theorem of Meusnier , we get :

$$x = N \cos\phi \qquad (3.8)$$

Comparing with (3.4) we get :

$$N = \frac{a\sqrt{1+e'^2}}{\sqrt{1+e'^2 \cos^2 \phi}} = \frac{c}{V} \qquad (3.9)$$

Eliminating c from (3.7) and (3.9) yields :

$$V^2 = \frac{N}{M} \qquad (3.10)$$

In subsequent sections we often need derivatives of V and η^2 as defined in (3.6) once and for all we calculate their derivatives with respect to ϕ :

$$\frac{d\eta^2}{d\phi} = e'^2\, 2\cos\phi(-\sin\phi) = -2\eta^2 \tan\phi \qquad (3.11)$$

We introduce :

$$t = \tan\phi \qquad (3.12)$$

And get :

$$\frac{d\eta^2}{d\phi} = -2\eta^2 t$$

$$\frac{d^2\eta^2}{d\phi^2} = -2(\eta^2 \sec^2\phi + \tan\phi \frac{d\eta^2}{d\phi}) = -2\eta^2(1-t^2) \quad \frac{d^3\eta^2}{d\phi^3} = 8\eta^2 t$$

Similarly starting from (3.6) we get :

$$\frac{dV}{d\phi} = \frac{1}{2V}\frac{d\eta^2}{d\phi} = \frac{1}{2V}(-2\eta^2 \tan\phi) = \frac{-\eta^2 t}{V}$$

$$\frac{d^2V}{d\phi^2} = \frac{-\eta^2}{V^3}(1-t^2+\eta^2)$$

$$\frac{d^3V}{d\phi^3} = \frac{t\eta^2}{V^5}(4+5\eta^2+\eta^4+3\eta^2t^2)$$

4. Length of a Meridional Arc

When dealing with conformal mapping equation we often need to Compute the length of the arc of a meridian between two points with latitudes ϕ_1 and ϕ_2 . From (3.7) we have :

$$B_{1,2} = \int_{\phi_1}^{\phi_2} M d\phi = c \int_{\phi_1}^{\phi_2} \frac{d\phi}{V^3} \qquad (4.1)$$

This integration leads to an elliptic integral of the second kind , from a computational point of view this is no useful path to follow . Instead we develop the denominator into a series:

$$V^{-3} = 1 - \frac{3}{2}e'^2\cos^2\phi + \frac{15}{8}e'^4\cos^4\phi - \frac{35}{16}e'^6\cos^6\phi + \ldots \qquad (4.2)$$

For the international ellipsoid 1924 with $c = 6399936.6081\text{m}$ and $e'^2 = 0.006768170$ the arc length B in meters from Equator $\phi_1 = 0$ to the latitude $\phi_2 = \phi$ in radians , becomes :

$$B_{1,2} = \int_{\phi_1}^{\phi_2} M = c\int_{\phi_1}^{\phi_2} \frac{d\phi}{V^3} = c\int_0^{\phi} V^{-3}\,d\phi = c\int_0^{\phi}(1 - \frac{3}{2}e'^2\cos^2\phi + \frac{15}{8}e'^4\cos^4\phi - \frac{35}{16}e'^6\cos^6\phi + \ldots) = c\int_0^{\phi}(1 - \frac{3}{2}e'^2(\frac{\cos 2\phi + 1}{2}) + \frac{15}{8}e'^4(\frac{\cos 2\phi + 1}{2})^2 - \frac{35}{16}e'^6 \times (\frac{\cos 2\phi + 1}{2})^3 + \ldots) = 6367654.492\phi - 16226.47269\sin 2\phi + 16.9744169\sin 4\phi - 0.022606653\sin 6\phi \tag{4.3}$$

Example 1:

The Faculty of Science in Misurata has $(\phi, \lambda) = (32^0 20' 52.09' N, 15^0 5' 36.91' E)$.

Find B?

ϕ In radians becomes : $\phi = 0.977384381\text{rad N}$.

$$B = 6367654.492\phi - 16226.47269\sin 2\phi + 16.9744169\sin 4\phi - 0.022606653\sin 6\phi = 3580366.348m$$

Example 2:

The Misurata university has $(\phi, \lambda) = (32^0 22' 22.37' N, 15^0 4' 41.68' E)$. Find B?

ϕ In radians becomes : $\phi = 0.565013354\text{rad N}$, then :

$$B = 6367654.492\phi - 16226.47269\sin 2\phi + 16.9744169\sin 4\phi - 0.022606653\sin 6\phi = 3583147.322m$$

Example 3:

The stadium in Misurata has $(\phi, \lambda) = (32^0 21' 47.97' N, 15^0 2' 45.60' E)$ Find B?

ϕ In radians becomes : $\phi = 0.564846578$ rad N, then :

$$B = 6367654.492\phi - 16226.47269\sin 2\phi + 16.9744169\sin 4\phi - 0.022606653\sin 6\phi = 3582087.667m$$

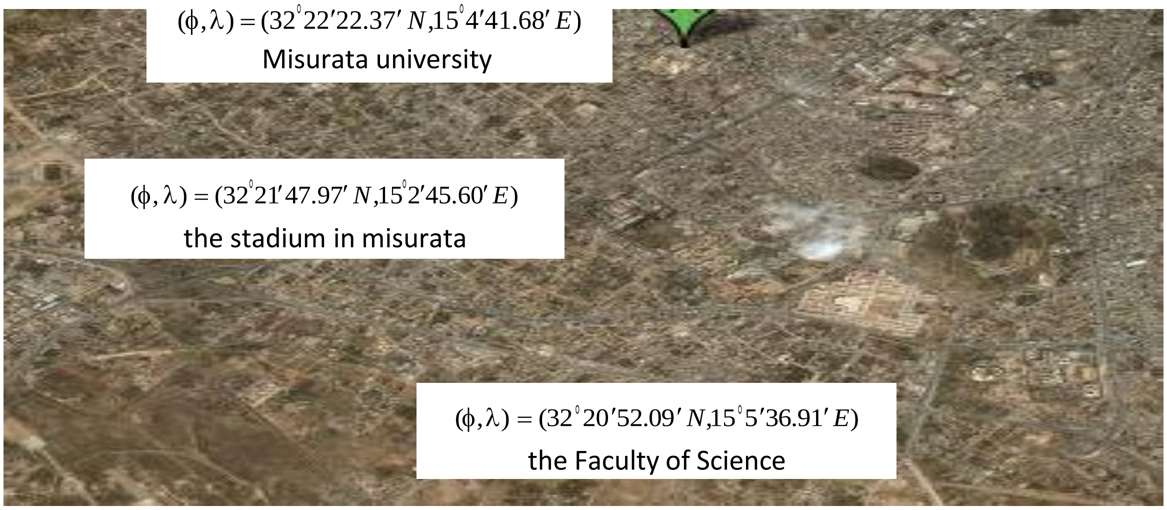

Figure 4. The geographic coordinate of the Misurata university, the Faculty of Science and the stadium in Misurata

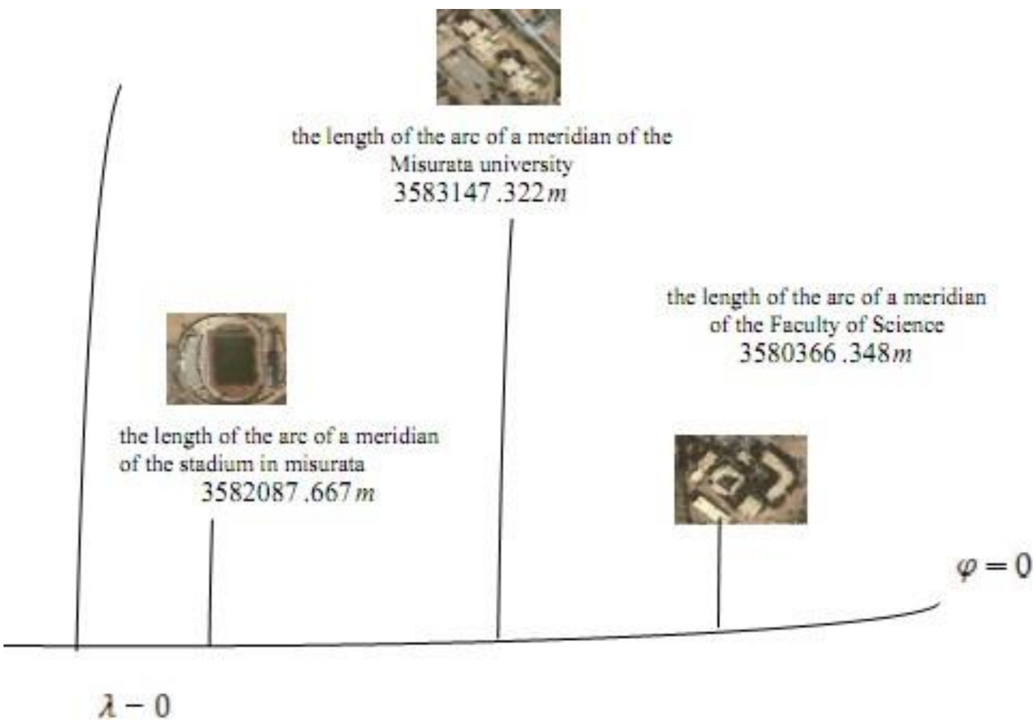

Figure 5. the length of the arc of a meridian of the last areas

5. The Gaussian Mapping of The Ellipsoid Onto The Plane

We use $2D$ Cartesian coordinate system with axis positive to the north and axis positive to the east, confer Figure 6. . In this section we use the unit of degrees as this is still the prevailing unit for geographical positions. The problem is to map conformally a geographical coordinate system (ϕ,λ) onto a plane Cartesian coordinate system (x, y) under the following conditions:

1. The central meridian having λ_0 = constant must be mapped onto the $x-$ axis of the plane system .
2. The plane image of the central meridian shall be mapped distance true .

Let the length differential ds which can be expressed in geographical coordinates (ϕ,λ) :

$$dS^2 = M^2 d\phi^2 + (N\cos\phi)^2 d\lambda^2 \qquad (5.1)$$

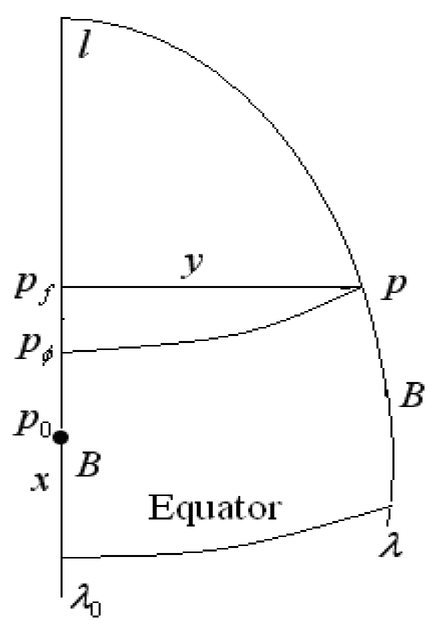

Figure 6. The Cartesian 2D plane with pertinent mapping of selected ellipsoidal quantities

We want to use complex function theory. This is only possible if $d\phi$ and $d\lambda$ have a factor in common .To obtain this we introduce the following substitution:

$$dq = \frac{M}{N\cos\phi}d\phi \tag{5.2}$$

The new variable q is called isometric latitude . let e, e' denote the eccentricity , then the explicit expression for q is :

$$q = \int_0^\phi \frac{M}{N\cos\xi}d\xi = \int_0^\phi \frac{c}{(1+e'^2\cos^2\xi)^{3/2}}\frac{\sqrt{1+e'^2\cos^2\xi}}{a\cos\xi\sqrt{1+e'^2}}d\xi$$

$$q = \int_0^\phi \frac{c}{a\sqrt{1+e'^2}(1+e'^2\cos^2\xi)\cos\xi}d\xi = \frac{c}{a\sqrt{1+e'^2}}\times \int_0^\phi \frac{d\xi}{(1+e'^2\cos^2\xi)\cos\xi} \tag{5.3}$$

By using fractions method ,then :

$$\frac{1}{\cos\xi(1+e'^2\cos^2\xi)} = \frac{A}{\cos\xi} + \frac{B\cos\xi + C}{1+e'^2\cos^2\xi} = \frac{A + Ae'^2\cos^2\xi + C\cos\xi + B\cos^2\xi}{\cos\xi(1+e'^2\cos^2\xi)}$$

$A=1, B=-e'^2$ and $C=0$.

$$q = \int_0^\phi \left(\frac{1}{\cos\xi} - \frac{e'^2\cos\xi}{1+e'^2\cos^2\xi}\right)d\xi = \int_0^\phi \sec\xi d\xi - \int_0^\phi \frac{e'^2\cos\xi}{1+e'^2\cos^2\xi}d\xi = \left[\ln(\sec\xi + \tan\xi)\right]_0^\phi - \int_0^\phi \frac{e'^2\cos\xi}{1+e'^2(1-\sin^2\xi)}d\xi \tag{5.4}$$

by using reparation method, then :

$$\int_0^\phi \frac{e'^2\cos\xi}{1+e'^2(1-\sin^2\xi)}d\xi = \int_0^\phi \frac{e'^2\cos\xi}{1+e'^2 - e'^2\sin^2\xi}d\xi$$

now let $e'\sin\xi = u \Rightarrow e'\cos\xi d\xi = du$ then :

$$\int_0^\phi \frac{e'^2\cos\xi}{1+e'^2-e'^2\sin^2\xi}d\xi = \int_0^\phi \frac{e'du}{1+e'^2-u^2} = e'\int_0^\phi \frac{du}{(\sqrt{1+e'^2}-u)(\sqrt{1+e'^2}+u)} \tag{5.5}$$

By using fractions method again :

$$\frac{1}{(\sqrt{1+e'^2}-u)(\sqrt{1+e'^2}+u)}=\frac{A}{(\sqrt{1+e'^2}-u)}+\frac{B}{(\sqrt{1+e'^2}+u)}$$

then :

$$1=2A\sqrt{1+e'^2}\Rightarrow A=\frac{1}{2\sqrt{1+e'^2}}=B$$

$$q=e'\int_0^{\phi}\frac{du}{(\sqrt{1+e'^2}-u)(\sqrt{1+e'^2}+u)}=$$

$$q=\left[\ln(\sec\xi+\tan\xi)\right]_0^{\phi}-\left[\frac{-e'\ln(\sqrt{1+e'^2}-e'\sin\xi}{\sqrt{1+e'^2}}\right]_0^{\phi}+\left[\frac{e'\ln(\sqrt{1+e'^2}+e'\sin\xi}{\sqrt{1+e'^2}}\right]_0^{\phi}$$

$$q=\ln\left((\sec\phi+\tan\phi)\left(\frac{\sqrt{1+e'^2}+e'\sin\phi}{\sqrt{1+e'^2}-e'\sin\phi}\right)^{-e'/\sqrt{1+e'^2}}\right) \tag{5.6}$$

by reparation $e=\dfrac{e'}{\sqrt{1+e'^2}}$ then :

$$q=\ln\left((\sec\phi+\tan\phi)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{e}\right) \tag{5.7}$$

Substituting dq in (5.2) we get:

$$dS^2=N^2\cos\phi^2(dq^2+d\lambda^2) \tag{5.8}$$

Now let l be the difference in longitude between the meridian containing the arbitrary point p and the central meridian λ_0 .

we describe a conformal mapping through the analytical function F :

$$x+iy=F(q+il) \tag{5.9}$$

The origin at the ellipsoid of revolution is taken as the intersection between the central meridian and equator . This gives the first boundary condition $(l = 0$ for $y = 0)$, condition 1 above:

$$x_{y=0} = F(q) \tag{5.10}$$

The second boundary condition tells that the meridian arc from equator to the point with isometric latitude q (corresponding to the geographical latitude ϕ) equals B , condition 2 above :

$$x_{y=0} = B = F(q) \tag{5.11}$$

Consequently the analytical function F measures the Meridional arc as a function of isometric latitude .We are not interested in F itself , but rather
its derivatives :

$$dx = dB = Md\phi = N\cos\phi dq \tag{5.12}$$

The last equality follows from (5.2) ,we divide by dq :

$$\frac{dx}{dq} = \frac{dB}{dq} = \frac{dF(q)}{dq} = F'(q) = N\cos\phi \tag{5.13}$$

According to the assumptions $l = \lambda - \lambda_0$ is small (a few degrees), but ϕ and q can take on any value between 0° to 90° .In order to work with small values for l and q we introduce an auxiliary point p_0 on the central meridian (in case of Denmark p_0 often has the value of 56° north latitude)in all further computations we work with differences in geographic and isometric latitude $\phi - \phi_0 = \Delta\phi$, $q - q_0 = \Delta q$. We insert these differences into (5.9)

$$x + iy = F(q_0 + \Delta q + il) \tag{5.14}$$

Next we expand F into a Taylor series around the point p_0 :

$$\begin{aligned} x + iy = F(q)_0 + F'(q)_0(\Delta q + il) + \frac{1}{2}F'(q)_0(\Delta q + il)^2 \\ + \frac{1}{6}F''(q)_0(\Delta q + il)^3 \ldots \end{aligned} \tag{5.15}$$

Remember from (5.11) that $F(q)_0 = B$ so :

$$x + iy = B_0 + (\frac{dB}{dq})_0 (\Delta q + il) + \frac{1}{2}(\frac{d^2B}{dq^2})_0 (\Delta q + il)^2 + \frac{1}{6}(\frac{d^3B}{dq^3})_0 (\Delta q + il)^3 ... \quad (5.16)$$

Let the coefficients in the series be $a_1, a_2, a_3, ...$ and we put $x - B = \Delta x$, then the approximated mapping equation is :

$$\Delta x + iy = a_1(\Delta q + il) + a_2(\Delta q + il)^2 + a_3(\Delta q + il)^3 ... \quad (5.17)$$

With :

$$a_1 = \left(\frac{dB}{dq}\right)_0 = \left(\frac{dB}{d\phi}\frac{d\phi}{dq}\right)_0 = \left(M\frac{N\cos\phi}{M}\right)_0 = N_0 \cos\phi_0$$

$$a_2 = \frac{1}{2}\left(\frac{d^2B}{dq^2}\right) = \frac{1}{2}\left(\frac{d}{d\phi}\frac{dB}{dq}\frac{d\phi}{dq}\right)_0 = \frac{1}{2}\left[\frac{d}{d\phi}\left(\frac{N^2\cos^2\phi}{M}\right)\right]$$

$$= \frac{1}{2}\left(\frac{-N^2}{M}2\cos\phi\sin\phi + \cos^2\phi\frac{2NMdN - N^2dM}{M^2}\right) =$$

$$\frac{1}{2}\left(\frac{-2N^2}{M}\cos\phi\sin\phi + \frac{2N}{M}\cos^2\phi dN - \frac{N^2}{M^2}\cos\phi dM\right)$$

then:

$$a_2 = -tN\cos^2\phi\left(V^2 + \eta^2\right) \quad (5.18)$$

Also, we get:

$$a_3 = \frac{-N}{6}\cos^2\phi\left(t\eta^2 + 2t^2V^2\cos\phi - V^2\cos\phi + 2t^2\eta^4\cos\phi - 2V^2\eta^2 + 8V^2t^2\eta^2\cos\phi\right) \quad (5.19)$$

Thus we have :

$$\left.\begin{aligned} a_1 &= N_0 \cos\phi_0 \\ a_2 &= -t_0 N_0 \cos^2\phi_0 \left(V^2 + \eta^2\right)_0 \\ a_3 &= \frac{-N_0}{6}\cos^2\phi_0 \left(t\eta^2 + 2t^2V^2\cos\phi - V^2\cos\phi + 2t^2\eta^4\cos\phi - \right. \\ &\left. 2V^2\eta^2 + 8V^2t^2\eta^2\cos\phi\right)_0 \end{aligned}\right\} \tag{5.20}$$

Separating the real and the imaginary parts in (5.17) we obtain the series for Δx and Δy. On the right side we have powers of Δq and l.

Still looking for simple formulas we introduce a trick ,instead of using the auxiliary point p_0 on the central meridian as expansion point we introduce another point of expansion , namely p_ϕ which also is on the central meridian and having the same latitude ϕ as the point to be mapped .Then follows that $\Delta\phi = \Delta q = 0$ and in the parentheses in (3.17) is only left il.

$$\begin{aligned} \Delta x + iy &= a_1 il - a_2 l^2 - a_3 il^3 + a_4 l^4 - a_5 il^5 + \ldots \\ x - B + iy &= a_1 il - a_2 l^2 - a_3 il^3 + a_4 l^4 - a_5 il^5 + \ldots \end{aligned} \tag{5.21}$$

separation into the real and the imaginary parts yields :

$$\left.\begin{aligned} x &= B - a_2 l^2 + a_4 l^4 + \\ y &= a_1 l - a_3 l^3 + a_5 l^5 \end{aligned}\right\} \tag{5.22}$$

The coefficients a_i shall be computed for latitude ϕ .The final mapping equations read :

$$\left.\begin{aligned} x &= B + tN\cos^2\phi(V^2 + \eta^2)l^2 + \ldots \\ y &= N\cos\phi l + \frac{N}{6}\cos^2\phi(t\eta^2 + 2t^2V^2\cos\phi - V^2\cos\phi + \\ &2t^2\eta^4\cos\phi - 2V^2\eta^2 + 8V^2t^2\eta^2\cos\phi)l^3 + \ldots \end{aligned}\right\} \tag{5.23}$$

Example 4:

The Misurata university has the geographic coordinate : $(\phi,\lambda) = (32^0 22' 22.37' N, 15^0 4' 41.68' E)$,the college of science has the geographic coordinate $(\phi,\lambda) = (32^0 20' 52.09' N, 15^0 5' 36.91' E)$ and the stadium has the geographic coordinate $(\phi,\lambda)=(32^{0}21'47.97'\,N,15^{0}2'45.60'\,E)$. Find the Cartesian coordinate ?

Solution :

From equations (5.23) ,we can get the Cartesian coordinates , such that :First : the Misurata university $(\phi,\lambda)=(32^{0}22'22.37'\,N,15^{0}4'41.68'\,E)$:

$B = 3583147.322m$, $\cos^2\phi = 0.84458145, \eta^2 = 0.004827856$

$V^2 = 1.004827856$, $N = 6384543.336$, $l = 0.263165011 rad$

$l^2 = 0.069255822 rad$, $l^3 = 0.018225709 rad$.

$$x = B + tN\cos^2\phi(V + \eta^2)l + \ldots = 3785031.05m$$

$$y = N\cos\phi l + \frac{N}{6}\cos^2\phi(t\eta^2 + 2t^2V^2\cos\phi - V^2\cos\phi + 2t^2\eta^4\cos\phi - 2V^2\eta^2 +$$
$$8V^2t^2\eta^2\cos\phi)l^3 + \ldots = 1416843.016m$$

Second : the college of science $(\phi,\lambda)=(32^{0}20'52.09'\,N,15^{0}5'36.91'\,E)$:

$\phi = 32.34780278^{0}$, $t = 0.6 \quad 3$, $\cos\phi = 0.84481572$,

$B = 3580366.348m$, c $^2\phi$ =o0.7 s 1 , $\eta^2 = 0.004830534$,

$V^2 = 1.0 \quad 0$, $N = 6384534.826$, $l = 0.2 \quad 6 \quad r \; 3$,

$l^2 = 0.0 \quad 6 \quad r \; 9$, $l^3 = 0.018281398 rad$.

$$x = B + tN\cos^2\phi(V + \eta^2)l + \ldots = 3782578.3m$$

$$y = N\cos\phi l + \frac{N}{6}\cos^2\phi(t\eta^2 + 2t^2V^2\cos\phi - V^2\cos\phi + 2t^2\eta^4\cos\phi - 2V^2\eta^2 +$$
$$8V^2t^2\eta^2\cos\phi)l^3 + \ldots = 1418651.803m$$

Third : the stadium $(\phi,\lambda)=(32^{0}21'47.97'\,N,15^{0}2'45.60'\,E) : \phi = 32.363325^{0}$,

$t = 0.633721767$, $\cos\phi = 0.844670735$, $B = 3 \quad .6 \quad m$,

$\cos^2\phi = 0.71346865$, $\eta^2 = 0.0 \quad 0 \quad 4$, $V^2 = 1.0 \quad 0 \quad 4$,

$N = 6384540.093 \quad , l = 0.2 \quad 6 \quad 2r \quad , l^2 = 0.0 \quad 6 \quad 8r \quad ,$
$l^3 = 0.0 \quad 1 \quad 8r \quad .$

$$x = B + tN\cos^2\phi(V + \eta^2)l + \ldots = 3783077.539m$$

$$y = N\,c \quad \phi l + \frac{N}{6}\mathrm{c} \quad {}^2\phi(t\eta^2 s + 2t^2V^2\,\mathrm{c\,s}\,\phi - V^2\,\mathrm{o} \quad \phi + 2t^2\eta^4\,\mathrm{c} \quad \phi - 2V^2\eta^2 +$$
$$8V^2t^2\eta^2\cos\phi)l^3 + \ldots = 1413963.80m$$

6. Transformation of Cartesian to Geographical Coordinates

The transformation consists of two steps .First we compute the isometric ellipsoidal coordinates from the Cartesian coordinates , next the isometric latitude is converted to geographical latitude .We change from Cartesian to isometric coordinates by using the inverse function f of F :

$$q + il = f(x + iy) \tag{6.1}$$

The boundary conditions for f are :

$$q_{l=0} = f(x) = f(B) \quad \text{and} \quad f'(x) = \frac{dq}{dB} = \frac{1}{N\cos\phi} \tag{6.2}$$

The right side of (6.1) can again be developed at the point p_0 and with $q - q_0 = \Delta q$ and $x - B_0 = \Delta x$ we get :

$$\Delta q + il = \left(\frac{dq}{dB}\right)_0(\Delta x + iy) + \frac{1}{2}\left(\frac{d^2q}{dB^2}\right)_0(\Delta x + iy)^2 + \frac{1}{6}\left(\frac{d^3q}{dB^3}\right)_0(\Delta x + iy)^3 + \ldots \tag{6.3}$$

Or introducing a set of coefficients b_i :

$$\Delta q + il = b_1(\Delta x + iy) + \frac{1}{2}b_2(\Delta x + iy)^2 + \frac{1}{6}b_3(\Delta x + iy)^3 \ldots \tag{6.4}$$

Explicitly the coefficients b_i are :

$$\left.\begin{aligned} b_1 &= \frac{1}{N_0 \cos\phi_0} \\ b_2 &= \frac{t}{2N_0^2 \cos\phi_0}(-4\eta^2 + V^2)_0 \end{aligned}\right\} \tag{6.5}$$

Again the series expansion takes place at the foot point $p_f = (\phi_f .0)$ of p_0 so $\Delta x = 0$ and the separation into the real and the imaginary parts yields :

$$\Delta q + il = b_1 iy + b_2 (iy)^2 + b_3 (iy)^3 + b_4 (iy)^4 + \ldots \tag{6.6}$$

$$\left.\begin{aligned} \Delta q &= -b_2 y^2 + b_4 y^4 - \\ l &= b_1 y - b_3 y^3 + b_5 y^5 - \end{aligned}\right\} \tag{6.7}$$

The coefficients b_i must be computed for ϕ_f .

We do not seek Δq but rather $\Delta\phi$ .The transition from Δq to $\Delta\phi$ also happens by means of a series in which we introduce $\Delta\phi = \phi - \phi_f$ and $\Delta q = q - q_f$. Then :

$$\phi + i\lambda = f(q + il) \tag{6.8}$$

The boundary conditions for f are :

$$\phi_{\lambda=0} = f(q) \quad \text{and} \quad \frac{N\cos\phi}{M} = \frac{d\phi}{dq} = \frac{df(q)}{dq} \tag{6.9}$$

Then :

$$\phi_0 + \Delta\phi + i\lambda = f(q_0 + \Delta q + il)$$

$$\phi_0 + \Delta\phi + i\lambda = f(q)_0 + f'(q)_0 (\Delta q + il) + \frac{1}{2} f'(q)_0 (\Delta q + il)^2 + \frac{1}{6} f''(q)_0 (\Delta q + il)^3 \ldots \tag{6.10}$$

$$\Delta\phi + i\lambda = f'(q)_0 (\Delta q + il) + \frac{1}{2} f'(q)_0 (\Delta q + il)^2 + \frac{1}{6} f''(q)_0 \times (\Delta q + il)^3 \ldots$$

$$\Delta\phi + i\lambda = d_1 (\Delta q + il) + \frac{1}{2} d_2 (\Delta q + il)^2 + \frac{1}{6} d_3 \times (\Delta q + il)^3 + \ldots$$

$$\Delta\phi = d_1 (\Delta q) + \frac{1}{2} d_2 ((\Delta q)^2 - l) + \ldots$$

$$\phi - \phi_f = d_1(\Delta q) + \frac{1}{2} d_2((\Delta q)^2 - l) + \dots \tag{6.11}$$

with

$$\left.\begin{aligned} d_1 &= \cos\phi_f (1+\eta^2{}_f) \\ d_2 &= -2tV\cos^2\phi(1-\eta^2) \end{aligned}\right\} \tag{6.12}$$

Insertion of Δq yields :

$$\phi - \phi_f = -b_2 d_1 y^2 + (b_4 d_1 + b_2^2 d_2) y^4 + \dots \tag{6.13}$$

Finally by inserting the coefficients b_i and d_i we get :

$$\left.\begin{aligned} \phi &= \phi_f - \frac{t_f}{2N_f{}^2}(-4\eta_f{}^2 + V_f{}^2)(1+\eta_f{}^2) y^2 + \dots \\ \lambda &= l = \frac{1}{N_f \cos\phi_f} y - b_3 y^3 + b_5 y^5 - \end{aligned}\right\} \tag{6.14}$$

Since $l = \lambda - \lambda_0$, and $\lambda_0 = 0$.

Example5:

From the last example find the geographic coordinates of the Misurata university and the college of science?

Solution :

From equation (6.14), we can get the geographic coordinates such $p_0 = 32^{\circ}$.

$\phi = 32^{\circ}$, $t = 0.624869351$, $\cos\phi = 0.848048096$, $N = 6384417.171$,

$\cos^2\phi = 0.71918$, $\eta^2 = 0.00486757$, $V^2 = 1.00486757$. Then

The Misurata university has : $(\phi, \lambda) = (31^{0} 59' 5.149' N, 15^{0} 42' 4.113' E)$.

The college of science has : $(\phi, \lambda) = (31^{0} 59' 5.009' N, 15^{0} 43' 16.27' E)$.

Which result is in agreement with the starting values , except for the influence of accuracy errors .

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