On Some Techniques of Treating Weakly Singular Integrals.

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Abstract

The weak singularity issue arises in the boundary integral equations either directly or as a result of reducing the hypersingularity to at worst weak singularity by using some regularization formulae. Here we show some analytic and numerical techniques to deal with the weak singularity phenomenon of the boundary integral equations. These powerful techniques include the logarithmic Gaussian quadrature, subtraction of singularity and coordinate transformations that are formulated in such a way that the singularity is removed.

Keywords: Boundary Element Method, Boundary Integral Equations, Weakly Singular Integrals, Hypersingular Integrals, Telles' Transformation, Gauss Quadrature.

1. Introduction

Boundary Element Method (BEM) is a powerful technique for obtaining an approximate solution for Partial Differential Equations (PDEs) that arise in scientific and engineering applications, such as elastodynamics, fluid dynamics, wave scattering, radiation and propagation. BEM is now a very well established and well documented technique; see Banerjee and Watson [1, 2], Brebbia [3, 4, 5, 6, 7, 8, 9] for an introduction and applications. Within the BEM the governing PDE can be transformed into an equivalent system of boundary integral equations (BIEs) directly through Green's second identity [10]. For a rigorous mathematical insight into BIEs theories and their abstract formulation using functional analysis see Atkinson [11], Mikhlin [12], Kress [13], Colton [14] and Hackbusch [15]. The mechanism of the BEM is to discretize the boundary into a number of elements to

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compute the integrals numerically over such elements. Thus the system of BIEs is converted into a linear system of algebraic equations which can be solved numerically. The implementation of the BEM is reliant on the accurate evaluation of boundary integrals. Owing to the use of the two-point (source and receiver) singular fundamental solution, the boundary integral formulations suffer from singularities. These integrals are dependent on the distance R between the source point (r') and receiver point (r) which varies in value as the integrals are taken around the boundary. If the distance $R \gg 0$ then the kernel of the BIE is regular, but if $R = 0$ then the BIEs exhibit singularities whose order is dependent on the type of kernel. Thus the general implementation of the BEM requires the computation of singular integrals with various types of singularities, such as weakly, strongly, and hypersingular of the order $O(\ln|R|), O\left(\frac{1}{R}\right)$ and $O(\frac{1}{R^2})$, respectively. Accurate and efficient computations of such integrals have made the BEM an efficient and generally well-conditioned numerical solution procedures. A great deal of research has been devoted to various ways of dealing with the singularity. Sladek and Sladek [16] gave a survey on the treatment of all types of singularities. There are several techniques to evaluate various types of singularities. In principle, there are two directions for regularization that are discussed in the literature. The first aims to remove all the non-integrable singularities analytically before any discretization is performed; such an analytic regularization is not entirely general and may be limited to certain problems. In contrast, the regularizations after discretization analyse each individual integrals by cancelling the divergent terms globally using a Galerkin or collocation formulation [16, 17]. This approach heavily requires the smoothness of the boundary [17]. The most widely used technique for regularization is the conversion of the hypersingular integral into a Cauchy Principal Value (CPV) integral. Hornberger et al. [18] implemented the BEM for two-dimensional magnetic billiards; they express the hypersingular operator as a special limit similar to the CPV integral. Then they use the asymptotic expression of the free-space Green function.

Krisnasamy et al. [19] dealt with the hypersingularity for the acoustic wave scattering in three dimensions. Their method is free of any discretization assumption; it demands only sufficient smoothness on the density function of the hypersingular function at the singularity point for the Taylor series expansion to be applied. Then a conversion by Stokes's theorem is carried out to reduce the order of the singularity by converting the surface integrals into line integrals. They used a regularization relationship, which reduces the hypersingularity to a weak singularity.

Some approaches [20, 21] represent the hypersingular integrals in terms of the Hadamard finite-part [22]. For instance, Bose [22] regularized the hypersingular integral that arises from acoustic scattering problem in two and three dimensions by using the Hadamard finite-part representation. Most of the techniques developed for regularization reduce the singularity to at worst weakly singular, or a complete regularization such as the singularity-free formula that is derived in [23]. Kutt [24, 25] developed a numerical approach to evaluate one-dimensional hypersingular integrals in finite part representation using Gaussian quadrature.

Here in this paper we outline some powerful methods to deal with the weak singularity. The focus will be on weakly singular integrals, so we will demonstrate these methods specifically to this type of integrals, though most of the presented methods below can be applied successfully for a higher order of singularity. These include the logarithmic Gaussian quadrature, subtraction of the singularity, and suitable variable transformations such as the Tanh rule and Telles transformation.

2. Logarithmic Gaussian Quadrature

A simple way to numerically approximate a regular (non-singular) integral is the standard Gaussian quadrature (GQ). In one-dimension it can be implemented as,

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{N} f(x_i)\,\omega_i \,, \tag{1}$$

where x_i and ω_i are the Gaussian quadrature points and the associated weighting functions, respectively, and N is the number of integration points. The integration range for the standard GQ is from -1 and 1. Thus one may need a linear transformation to accommodate general integration limits from a to b, as follows,

$$\int_a^b f(x)dx = \left(\frac{b-a}{2}\right)\int_{-1}^{1} f\left[\frac{b+a}{2} + (\frac{b-a}{2})\zeta\right] d\zeta.$$

GQ evaluates the integrand at a given number of points called the Gaussian coordinates, then the function is multiplied by a weight function and the sum is taken to approximate the integral. A large number of Gaussian points leads to a better accuracy. The standard GQ is not sufficiently accurate for weakly singular (logarithmic) integrals, even with a large number of Gaussian points. The logarithmic Gaussian quadrature is designed to treat such integrals. This evaluates the integral of a function $f(x)$ over the range from 0 and 1 numerically as,

$$\int_0^1 f(x)\ln\left(\frac{1}{x}\right) dx \approx \sum_{i=1}^{N} f(x_i)\omega_i \, , \tag{2}$$

where the x_i and ω_i are the logarithmic Gaussian quadrature points and the associated weighting functions, respectively. It is noteworthy to mention that the integral is taken over the limits from 0 to 1. Therefore, one needs to do a linear transformation in order to accommodate general limits. We should emphasise that, for logarithmic Gaussian quadrature the Gaussian points and weighting functions are different from those for the standard Gaussian quadrature. The Gaussian quadrature points and the associated weighting functions for standard and logarithmic GQ for different values of N are tabulated in the textbook by Stroud and Secrest [26], and can also be found in [27].

Next, we will present another approach of regularization where one can subtract the singular part, and then evaluate the reminder (non-singular term) using Gaussian type quadrature whereas the singular part is integrated using analytic integration formula.

3. Singularity Subtractions

In principle, this technique can be used to evaluate all type of singular integrals [28]. The core idea of this technique is to remove the singular terms from the singular integrand leaving a regular integrand that can be integrated using standard Gaussian quadrature. Then the singular terms is included back in by an additional singular integral which can be computed analytically.

Here we will demonstrate the subtraction of singularity technique for weakly singular integral, though it can be used even for hypersingular integrals [29, 30].

Example: Within the BEM formulation for the Helmholtz equation, it is often to encounter the logarithmic singularity which requires a regularization. Such a logarithmic singularity comes from the integral of the free-space Green function $G_0(q, \beta; k)$ which is defined as,

$$G_0(q, \beta; k) = \frac{i}{4} H_0^{(1)}(k|q - \beta|),$$

where $H_0^{(1)}(k|q - \beta|)$ is the Hankel function of zeroth order [31] defined as,

$$H_0^{(1)}(k|q - \beta|) = J_0(k|q - \beta|) + iY_0(k|q - \beta|).$$

$J_0(k|q - \beta|)$ and $Y_0(k|q - \beta|)$ are the Bessel functions of the first and second kinds respectively, and defined as,

$$J_0(k|q - \beta|) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{k|q - \beta|}{2}\right)^{2k},$$

and,

$$Y_0(k|q - \beta|) = \frac{2}{\pi}\left(\ln\left(\frac{k|q - \beta|}{2}\right) + \gamma\right) J_0(k|q - \beta|) - \frac{2}{\pi}\sum_{k=0}^{\infty} a_k \frac{(-1)^k}{(k!)^2}\left(\frac{k|q - \beta|}{2}\right)^{2k},$$

where $a_k = \sum_{m=1}^{k} 1/m$ and $\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n} \frac{1}{m} - \ln n\right) = 0.57721 \ldots.$

The integral of $Y_0(k|q - \beta|)$ becomes logarithmically singular when q and β are in the same boundary element. This weak singularity can be treated by expanding the Hankel function and integrating out the logarithmically divergent term explicitly as,

$$\int G_0(q,\beta;k)dq = \frac{i}{4}\left\{\int\left[H_0^1(k|q-\beta|) - \frac{2i}{\pi}\ln\left(\frac{k|q - \beta|}{2}\right)dq\right] + \int \frac{2i}{\pi} ln\left(\frac{k|q - \beta|}{2}\right)dq\right\}. \quad (3)$$

This integral is weakly singular when $|q - \beta| = 0.$ The first integral in the right hand side of equation (3) is regular and can be integrated numerically using standard Gaussian quadrature. However, the second integral in the right hand side of equation (3) can be integrated analytically using the formula,

$$\int \ln|z|\, dz = z\,[ln|z| - 1]\,.$$

Thus, we have demonstrated that the Singularity subtraction procedure can be very useful in treating the weakly singular integrals. Next, we will discuss another approach for regularization that is based on a suitable coordinate transformation whose Jacobian smoothens out the singularity.

4. Coordinate Transformations

One of the best known coordinate transformations for treating weak singularities is attributed to Telles [32]. It is an elegant and simple way to deal with singular and nearly singular integrals. This transformation is performed in such a way that its Jacobian weakens or cancels out the singularity. Then the resulting integral can be evaluated by standard Gaussian type quadrature, such as Gauss-Legendre quadrature [27]. This transformation will be presented below where we will present some formulae.

Another efficient transformation to deal with weakly singular integrals is the tanh rule [33, 34, 35]. It works in such a way that the Jacobian decreases rapidly at the endpoints of the integration limits as described next. Furthermore, Hayami [36, 37] developed another transformation which needs to be implemented in polar coordinates.

3.1 Tanh Rule

This rule requires dividing the integral into a sum of two integrals at the singularity point as,

$$I = \int_{\alpha}^{\beta} g(y)dy,$$

where the integrand $g(y)$ has a singularity at α or β. The core idea of this rule is that if the singular integrand does not vanish on the endpoints of the integral, then one needs to think of a transformation $g = g(w)$ that ignores the endpoints singularity completely .

Therefore, we choose the transformation $g(w)$ such that the factor $\frac{dy}{dw}$ decreases rapidly (goes to zero) at the endpoints of the interval. Then a simple trapezoidal rule can be used to evaluate the non-singular integral giving extremely accurate results. Tanh Rule was introduced by Schwartz [34] and has become known as the Tanh rule; it is based on the following variable transformation,

$$I = \int_{-\infty}^{+\infty} g[y(w)]\frac{dy}{dw}\,dw, \tag{4}$$

where,

$$y = \left(\frac{\beta+\alpha}{2}\right) + \left(\frac{\beta-\alpha}{2}\right)\tanh(\omega)\,, \qquad -\infty < \omega < \infty\,,$$

so,

$$\frac{dy}{dw} = \left(\frac{\beta-\alpha}{2}\right)\operatorname{sech}^2(\omega) = \frac{2}{(\beta-\alpha)}(\beta-y)(y-\alpha)\,,\ \ -\infty < \omega < \infty,\quad \alpha < y < \beta.$$

Clearly, one can see that the factor $\frac{2}{(\beta-\alpha)}(\beta-y)(y-\alpha)$ vanishes at the singularity point α or β which corresponds to the factor $\operatorname{sech}^2(\omega)$, which vanishes as $w \to \pm\infty$. So the integral (4) is transformed to the following non-singular integral,

$$I = \int_{-\infty}^{\infty} g\left[\left(\frac{\beta+\alpha}{2}\right) + \left(\frac{\beta-\alpha}{2}\right)\tanh(\omega)\right]\left(\frac{\beta-\alpha}{2}\right)\operatorname{sech}^2(\omega)\,dw. \tag{5}$$

One can notice that the term $\operatorname{sech}^2(\omega)$ decreases sharply as $\to \pm\infty$, this explains the efficiency of this technique. Since the term $\operatorname{sech}^2(\omega) \to 0$ as $w \to \pm\infty$, we can truncate the integral (5) at the values $\pm w_{max}$, hence one obtains the following non-singular integral which can be computed using Gaussian quadrature scheme,

$$I \approx \int_{-w_{max}}^{+w_{max}} g\left[\left(\frac{\beta+\alpha}{2}\right) + \left(\frac{\beta-\alpha}{2}\right)\tanh(\omega)\right]\left(\frac{\beta-\alpha}{2}\right)\operatorname{sech}^2(\omega)\,dw.$$

So we truncate the infinite limits of the integral to a finite value of w_{max}, this leads to a trimming error. However the larger value of w_{max} , the smaller trimming error will be.

Schwartz [36] gives an estimate of the trimming and the discretization errors.

Next we will discuss the best known coordinate transformations and we will present some formulae and illustrative examples.

3.2 Telles' Transformation

The Telles transformation [32] is a popular variable transformation method to treat the weakly singular integral. It was originally designed to compute one-dimensional integrals with a logarithmic singularity. The Jacobian of such transformation merely weakens or cancels out the singularity. This approach uses a quadratic or cubic variable transformation, depending on whether the singular point is lying at the end of the element (integration domain) or within the element, respectively. Consider the integral,

$$I = \int_{-1}^{1} f(x)dx ,$$

where $f(x)$ is a weakly singular function at one of the extremities of the integral. This means that the singularity point x_0 satisfies the condition $|x_0| = 1$. We can use the following quadratic transformation for the integral I,

$$I = \int_{-1}^{1} f\left[(1-\zeta^2)\frac{x_0}{2} + \zeta\right](1-\zeta x_0)d\zeta , \qquad (6)$$

where ζ is the new coordinate variable. The key point is that the Jacobian of the transformation $(1-\zeta x_0)$ vanishes at the singularity point x_0. Thus this transformation cancels out the singularity and produces a regular integral. Then the resulting non-singular integral can be evaluated numerically by standard Gaussian quadrature schemes. To demonstrate the efficiency of Telles' transformation, we give the following examples.

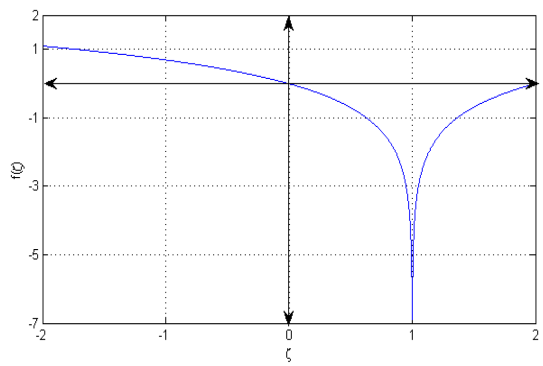
Example 1: Evaluate the following integral:

$$I = \int_{-1}^{1} ln|1-\zeta|d\zeta .$$

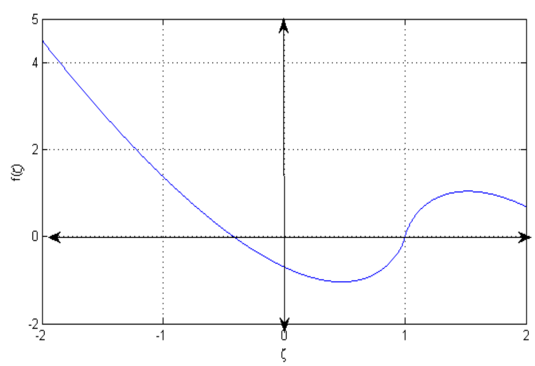
This integral is weakly singular at the point $\zeta = 1$ as shown in figure 1(a), so the integrand needs regularization. Thus by using Telles' transformation given by equation (6) where we substitute the singularity point $x_0 = 1$, this logarithmically singular integral can be transformed to,

$$I = \int_{-1}^{1} \ln\left|1 - \left(\frac{(1-\zeta^2)}{2} + \zeta\right)\right| (1-\zeta)d\zeta.$$

This regular integral can be computed by using the standard Gaussian quadrature given by equation (1). So one can sees that the weakly singular integrand $ln|1-\zeta|$ is regularized and the singularity is removed by the Jacobian $(1-\zeta)$ as shown in figure 1(b).



(a) Weakly singular integrand f(ζ)=ln (1- ζ)



(b) Regularized integrand $\mathbf{f(\zeta) = \ln\left|1 - \left(\frac{(1-\zeta^2)}{2} + \zeta\right)\right| (1-\zeta)}$

Figure 1 Regularization of the weakly singular integrand f(ζ)=ln (1- ζ) by Telles' transformation which transforms the integrand to $\mathbf{f(\zeta) = \ln\left|1 - \left(\frac{(1-\zeta^2)}{2} + \zeta\right)\right| (1-\zeta)}$

If the singularity point satisfies the condition $|x_0| < 1$, then one needs to partition the integral into two integrals at the singularity point and then employ the transformation (6) for each integral. However, Telles introduced another cubic transformation for the case that the singularity point lays within the element, that is $|x_0| < 1$. Hence one needs to use the following non-linear transformation,

$$I = \int_{-1}^{1} f\left[\frac{(\zeta - \acute{\zeta})^3 + \acute{\zeta}(\acute{\zeta}^2 + 3)}{1 + 3\acute{\zeta}^2}\right] \frac{3(\zeta - \acute{\zeta}^2)}{1 + 3\acute{\zeta}^2}\, d\zeta, \qquad (7)$$

where $\acute{\zeta}$ is given as

$$\acute{\zeta} = \sqrt[3]{\gamma^* x_0 + |\gamma^*|} + \sqrt[3]{\gamma^* x_0 + |\gamma^*|} + x_0 \ ,$$

and γ^* is given as

$$\gamma^* = {x_0}^2 - 1.$$

To demonstrate the efficiency of the cubic transformation, we give the following example.

Example 2: Evaluate the following integral:

$$I = \int_{-1}^{1} ln|x|dx \ .$$

This integral is weakly singular at the point $x = 0$ which is located within the integration range, so one needs to use the cubic transformation (7) where $x_0 = 0$ as follows,

$$\gamma^* = -1 \quad \text{and} \quad \acute{\zeta} = \sqrt[3]{0+1} + \sqrt[3]{0-|1|} = 0,$$

So the integral can be expressed as,

$$I = \int_{-1}^{1} ln|x|dx = \int_{-1}^{1} ln\left|\frac{\zeta^3 + 0}{1+0}\right| 3\zeta d\zeta = 3\int_{-1}^{1} \zeta \ln|\zeta^3|\, d\zeta \ .$$

Therefore, one can easily notice that the Jacobian ζ cancels the singularity and vanishes at the singularity point $\zeta = 0$. Then the resulting regular integral can be computed using Gaussian quadrature. It is noteworthy to mention that Telles' transformation concentrates the Gauss points around the singularity point leading to a high accuracy with less number of Gauss points [32]. This makes Telles' transformation a very efficient technique to treat the weak singularity.

5. Conclusion

To summarize, we have shown some powerful techniques to treat weakly singular integrals, though some of them can also be used to treat higher order of singularity. Some of these techniques are numerical such as quadrature schemes which use certain integration points and weights that account for the singularity. The others are analytic such as the subtraction of singularity and the coordinate's transformation, however the analytic methods need sometimes to be followed by quadrature schemes to improve the accuracy. We have shown some examples confirming the efficiency of these techniques. As it has been addressed in section (2), the logarithmically singular integral can be evaluated using the logarithmic Gaussian quadrature. Nevertheless, using Telles' transformation first, and then using the Gaussian quadrature greatly improves the accuracy and gives better results than logarithmic Gaussian quadrature [32].

6. References

[1] P.K. Banerjee. The boundary element method in engineering. McGraw-Hill Publications, London, 1994.
[2] P.K. Banerjee, J. Watson. Development in boundary element methods. Elesvier Science Publishing Co., New York, 1986.
[3] C.A. Brebbia. Topics in boundary element research, Vol: 1 Basic principles and applications. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.
[4] C.A. Brebbia. Topics in boundary element research, Vol: 2 Time dependent and vibration problems. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.
[5] C.A. Brebbia. Topics in boundary element research, Vol: 3 Computational aspects. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1987.
[6] C.A. Brebbia, J.C.F. Tells, L.C. Wrobel. Boundary element techniques. Springer-Verlag, Berlin and New York, 1984.
[7] C.A. Brebbia, J. Dominguez. Boundary elements: An introduction course. Computational Mechanics Publications, McGraw-Hill Book Company, Southampton, 1992.
[8] C.A. Brebbia, J. Dominguez. Boundary elements methods for potential problem. App. Math. Modell., **1**:372, 1977.
[9] C. A. Brebbia, S. Walker. The boundary element techniques in engineering. Newnas-Butterworths, London, 1979.
[10] G. George. An essay on the application of mathematical analysis to the theories of electricity and magnetism. Printed for the Author by T. Wheelhouse, Nottingham, 72 pages, 1828.
[11] K.E. Atkinson. The numerical solution of integral equations of the second kind, Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, 1997.
[12] S.G. Mikhlin. Integral equations. Pergamon Press, Oxford, 1957.

[13] D. Colton, R. Kress. Integral Equation Methods in Scattering Theory. John Wiley and Sons, New York, 1983.

[14] R. Kress. Linear integral equations. Applied mathematical sciences; Vol. 82. Springer--Verlag, New York Inc., 2nd edition, 1999.

[15] W. Hackbusch. Integral equations: Theory and numerical treatment. *International Series of Numerical Mathematics (ISNM)*, Vol **120**, 1995.

[16] V. Sladek, J. Sladek. Singular integrals in boundary element methods. Computational Mechanics Publications, Southampton, 1998.

[17] M. Tanaka, V. Sladek, J. Sladek. Regularization techniques applied to boundary element methods. *Applied Mechanics Reviews*, **47**(10):457-499, 1994.

[18] K. Hornberger, U. Smilansky. The boundary integral method for magnetic billiards. *J. Phys. A*, **33**:2829-2855, 2000.

[19] G. Krisnasamy, L.W. Schmerr, T.J. Rudolphi, F.J. Rizzo. Hypersingular boundary integral equations: Some applications in acoustics and elastic wave scattering. *ASME Journal Applied Mechanics*, **57**:404-414, 1990.

[20] S.K. Bose. Finite part representations of hypersingular equation of acoustic scattering and radiation by open smooth surfaces. *Proc. Indian. Acad. Sci.*, **106**(3):271-280, 1996.

[21] P.A. Martin, F.J. Rizzo. Hypersingular integrals: How smooth must the density be? *International Journal for Numerical Methods in Engineering*, **39**(3):687-704, 1996.

[22] J. Hadamard. Lectures on Cauchy's problem in linear partial differential equations. Yale University Press, New Haven, 1923.

[23] W.L. Meyer, W.A. Bell, and B.T. Zinn. Boundary integral solutions of three dimensional acoustic radiation problems. *Journal Sound and Vibration*, **59**(2):245--262, 1978.

[24] H.R. Kutt. The numerical evaluation of principle value integrals by finite part integration. *Numer. Math*., **24**:205-210, 1975.

[25] H.R. Kutt. Quadrature formula for finite part integrals. *CSIR Special Report WISK*, 178, Pretoria, National Research Institute for Mathematical Sciences, P.O. Box 395, Pretoria 0001, South Africa, 1975.

[26] A.H. Stroud, D. Secrest. Gaussian quadrature formula. Printice-Hall, New York, 1966.

[27] A. A. Becker. The boundary element method in engineering: A complete Course. McGraw-Hill Book Company Europe, London, 1992.

[28] M. Guiggiani and P. Casalini. Direct computation of Cauchy principal value integrals in advanced boundary elements. *International Journal for Numerical Methods in Engineering*, **24**:1711–1720, 1987.

[29] M. Guiggiani. Hypersingular formulation for boundary stress evaluation. Engineering Analysis with Boundary Elements, 13(2):169–179, 1994.

[30] M. Guiggiani, G. Krishnasamy, T.J. Rudolphi, and F.J. Rizzo. A general algorithm for the numerical-solution of hypersingular boundary integral equations. *Journal of Applied Mechanics-Transactions of the ASME*, **59**:604–614, 1992.

[31] M. Abramowitz, A. Stegun. Handbook of mathematical functions. New York: Dover Publications, Inc., 9th edition, 1968.

[32] J.C.F. Telles. A self-adaptive coordinate transformation for efficient numerical evaluation of general boundary element integrals. *International journal for numerical methods in engineering*, **24**(4):959-973, 1989.

[33] P.J. Davis, P. Rabinowitz. Methods of numerical integration. Academic Press, Inc., Orlando, Florida, 2nd edition, 1984.

[34] C. Schwartz. Numerical integration of analytic functions. *Journal of Computational Physics*, **4**:19-29, 1969.

[35] G. A. Evans, R. C. Forbes and J. Hyslop. The Tanh Transformation for Singular Integral. *International Journal of Computer Mathematics*, **15**: 339-358, 1984.
[36] K. Hayami. A projection transformation method for nearly singular surface boundary element integrals. PhD thesis, Wessex Institute of Technology, Southampton, 1991.
[37] K. Hayami. A new coordinate transformation method for singular and nearly singular integrals over general curved boundary elements. *Boundary elements IX*, **1**:375-399, 1987.