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Oscillation Criteria for a Class of Second-Order Nonlinear Difference Equations

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Abstract

In this paper, we are concerned with the oscillation of a class of second- order non-linear difference equations. By using the Riccati technique some new oscillation criteria are established, therefore, we generalize and extend a number of existing oscillation criteria. An example is also given to illustrate our results.

Keywords: Différences équations, Oscillation, Ricati technique.

1. Introduction

This paper is concerned with the oscillation of the solutions of the second-order non-linear difference equation

$$\Delta\left(k(n, x_n, \Delta x_n)\right) + q_n \varphi\left(g(x_{n+1}), k(n+1, x_{n+1}, \Delta x_{n+1})\right) = 0, n = 0,1, \qquad (E)$$

Where Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$ for any sequence $\{x_n\}$ of real numbers, $\varphi \in C(\mathbb{R}^2, \mathbb{R})$ with $u\varphi(u,v) > 0 \forall u \neq 0, \frac{\partial \varphi(u,v)}{\partial v} \leq 0 \forall u \neq 0$ and $v \in \mathbb{R}$ and $\varphi(\lambda u, \lambda v) = \lambda\varphi(u,v)$ where $\lambda > 0, g \in C(\mathbb{R}, \mathbb{R})$ with $xg(x) > 0\ \forall x \neq 0$, and $g(u) - g(v) = g_1(u,v)(u-v)^\delta$ for $u, v \neq 0, \delta > 0$ is the ratio of odd positive integers, $g_1(u,v) \geq 0$ and $g(u) \geq g(v)$ iff $u \geq v, k \in C^1(\mathbb{N} \times \mathbb{R}^2, \mathbb{R})$ with $wk(u,v,w) > 0 \forall w \neq 0$, and $\{q_n\}_{n=0}^{\infty}$ is a sequence of real values.

A solution of (E) is a nontrivial real a sequence $\{x_n\}$ satisfying Equation (E) for $n \geq 0$. A solution $\{x_n\}$ of (E) is said to be oscillatory if is neither eventually positive nor eventually negative, otherwise it is nonoscillatory Equation (E)issaid to be oscillatory if all its solutions are oscillatory.

There are a great number of papers devoted to particular cases of equation (E) such as

$$\Delta(r_n(\Delta x_n)^{r}) + q_n x_{n+1}^{r} = 0, n = 0,1, ...,$$

$$\Delta(r_n \Delta x_n) + q_n g(x_{n+1}) = 0, n = 0,1, ...,$$

and

$$\Delta(r_n \psi\,(x_n)\Delta x_n) + q_n g(x_{n+1}) = 0, n = 0,1, ...,$$

See for example ([1-4, 6, 7,9-26]) and references cited therein.

For the oscillation of

$$\Delta(r_n \psi\,(x_n) f\,(\Delta x_n)) + q_n \varphi(g(x_{n+1}), r_n + 1\psi(x_{n+1})\, f(\Delta x_{n+1})) = 0, n = 0,1, ...,$$

(E_1)

Where ψ and f are containuous functions on $\mathbb{R}$with $\psi\,(x) > 0$ and $xf\,(x) > 0$ for all $x \neq 0$, and $\{r_n\}_{n=0}^{\infty}$ is sequence of positive real numbers.

For the equation (E_1), E. M. Elabbasy and Sh. R. Elzeiny [5: Theorem 2.1], proved that, if there exist a constant $c_1 \in \mathbb{R}_+$ such that

$$\Phi(m) = \int_0^m \frac{dv}{\varphi(1,v)} \geq -c_1 \text{ for every } m \in \mathbb{R}, \qquad (1.1)$$

and

$$\limsup_{t\to\infty} \sum_{i=n0}^{n-1} q_i = \infty. \qquad (1.2)$$

Then every solution of equation (E) oscillates.

Also ,they [5: Lemma 2.2], proved that, if $f(y) = y^{r}$, where r is the ratio of odd positive integers, and there exist positive integers N_0 and $N_1, N_1 \geq N_0$ such that

$$\sum_{i=N_0}^{\infty} q_i \geq 0 \; and \; \sum_{i=N_1}^{\infty} q_i > 0 \forall\, N_1 \geq N_0, \qquad (1.3)$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{r_n}\right)^{\frac{1}{r}} = \infty, \tag{1.4}$$

The function $(\frac{\psi}{g})$ is nonincreasing for all $x \neq 0$, (1.5)

$F(u) - F(v) = F_1(u,v)(u - v), for\ u,v \neq 0, F_1(u,v) < 0$ and

$$F(u) \geq F(v) \text{iff u } \leq \text{v, where } F(\omega) = \varphi(1,\omega), \tag{1.6}$$

And $\{x_n\}$ is a non-oscillatory solution of equation (E_1) such that $x_n > 0$ for all $n \geq N$, then there exists an integer $N \geq N_1$ such that $\Delta x_n > 0$ for all $n \geq N$.

Our objective here is to proceed further in this direction to obtain some new sufficient conditions for oscillation of solutions of equation(E) and some of our results obtained by implying and extending those in ([1-7, 9-26)].

2. Main Results

For strain hardening material, the yield surface must change in some way so that an increase in
Theorem 2.1. Assume that (1.1) and (1.2) hold. Then every solution of equation (E)oscillates.

Proof: suppose to the contrary that $\{x_n\}$ is a nonoscillatory solution of (E).

Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive solution of (E) such that $x_n > 0, n \geq n_0$.Define the sequence $\{\omega_n\}$ by

$$\omega_n = \frac{k(n, x_{n,} \Delta x_{n)}}{g(x_n)}, n \geq n_{0.}$$

Then , for all $n \geq n_0$, we have

$$\Delta\omega_n = \frac{\Delta(k(n, x_n, \Delta x_n))}{g(x_{n+1})} - k(n, x_n, \Delta x_n)\frac{\Delta\big(g(x_n)\big)}{g(x_n)g(x_{n+1})}.$$

This and (E) imply

$$\Delta\omega_n = -\varphi(1, \omega_{n+1})q_n - k(n, x_n, \Delta x_n)\frac{g_1(x_{n+1,}x_n)(\Delta x_n)^{\delta}}{g(x_n)g(x_{n+1})}.$$

Hence, for all $n \geq n_0$, we obtain

$$\Delta\omega_n \leq -\varphi(1, \omega_{n+1})q_n.$$

Or

$$\varphi(1,\omega_{n+1})q_n \le -\Delta\omega_n, n \ge n_0.$$

Dividing this inequality by$\varphi(1,\omega_{n+1}) > 0$, We obtain

$$q_n \le -\frac{\Delta\omega_n}{\varphi(1,\omega_{n+1})}, n \ge n_0. \tag{2.1}$$

Summing (2.1) from n_0 to $n-1$, we have

$$\sum_{m=n_0}^{n-1} q_m \le -\sum_{l=n_0}^{n-1} \frac{\Delta\omega_l}{F(\omega_{l+1})}, \qquad \text{where} \quad F(\omega_n) = \varphi(1,\omega_n). \tag{2.2}$$

Define $\delta\,(t) = \omega_l + (t-l)\,\Delta\omega_l, t \in [l, l+1]$. Then we have one of the following two cases

Case (1): If $\Delta\omega_l \ge 0, then\ \omega_l \le \ \delta(t) \le \omega_{l+1.}$ Thus,in view of the definition of the functionφ,we get

$$\frac{\Delta\omega_l}{F(\omega_l)} \le \frac{\delta'(t)}{F\left(\delta(t)\right)} \le \frac{\Delta\omega_l}{F(\omega_{l+1})}. \tag{2.3}$$

Case (2): If $\Delta\omega_l \le 0, then\ \omega_{l+1} \le \ \delta(t) \le \omega_l$. So we can directly obtain (2.3).

Now, by (2.2) and (2.3), we get

$$\sum_{m=n_0}^{n-1} q_m \le -\int_{n_0}^{n} \frac{d\big(\delta(t)\big)}{F\left(\delta(t)\right)} = -\int_{\delta(n_0)}^{\delta(n)} \frac{du}{\varphi(1,u)} = -\big[\Phi\big(\delta(n)\big) - \Phi\big(\delta(n_0)\big)\big]$$

$$\le \ c_1 + \Phi\big(\delta(n_0)\big) = c_1 + \Phi(\omega_{n0}). \tag{2.4}$$

Taking the limit superior on both sides for (2.4), we obtain

$$\lim_{t\to\infty} \sum_{i=n_0}^{n-1} q_i < \infty,$$

Which contradicts (1.2). Hence, the proof is completed.

Example 2.1: Consider the difference equation

$$\Delta\big((n^2 + x_n^2 - 4x_n\Delta x_n + 4(\Delta x_n)^2)\Delta x_n\big) + (1 + 2(-1)^n)\varphi(u, v) = 0, n \geq 1. \quad (2.5)$$

Here, $k(n, x_n, \Delta x_n) = (n^2 + x_n^2 - 4x_n\Delta x_n + 4(\Delta x_n)^2)\Delta x_n, q_n = 1 + 2(-1)^n$,

And $\varphi(u, v) = ue^{-\frac{v}{u}}$, where $u = g(x_{n+1}) = x_{n+1}^3$,and

$$v = ((n + 1)^2 + x_{n+1}^2 - 4x_{n+1}\Delta x_{n+1} + 4(\Delta x_{n+1})^2)\Delta x_{n+1}.$$

All conditions of Theorem 2.1 are satisfied, and hence, all solutions of equation (2.5) are oscillatory.

Note that the Results of E.M. Elabbasy and sh. R. Elzeiny [5] cannot be applied to (2.5).

Theorem 2.2: Assume that $k(n, x, y) \geq byr_n \; \forall \; y \in \mathbb{R}$ and for some constant $b > 0$. Furthermore, suppose that

$$\lim_{|\omega|\to\infty} inf \varphi\,(1, \omega) = c > 0, \quad (2.6)$$

$$\int_0^{\pm\varepsilon} \frac{du}{g(u)} < \infty \forall \varepsilon > 0, \quad (2.7)$$

$$\lim_{n\to\infty} \sup \sum_{m=n_0}^{n-1} \frac{1}{r_m} < \infty, \quad (2.8)$$

And

$$\lim_{n\to\infty} \sup \sum_{m=n_0}^{n-1} \left(\frac{1}{r_m} \left(\sum_{i=n_0}^{m-1} q_i \right) \right) = \infty \quad (2.9)$$

Then every solution of equation (E) oscillates.

Proof: Suppose to the contrary that $\{x_n\}$ is a nonoscillatory solution of (E).

Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive solution of (E) such that $x_n > 0, n \geq n_0$. Define the sequence $\{\omega_n\}$ as in the proof of the pervi- ous theorem. Following the same procedures, we get of Theorem (2.1). Now, we have one of the fol- lowing two cases:

Case (1): If $\Delta\omega_n \geq 0, then \; \omega_{n+1} \geq \omega_n \geq \omega_{n_0}$.

In view of the definition of the function φ, and the condition (2.6), we get

$$-\frac{\Delta\omega_n}{F(\omega_{n+1})} \le -\frac{\Delta\omega_n}{F\,(\omega_{n0})}, n \ge n_{0.} \tag{2.10}$$

Case (2): If $\Delta\omega_l \le 0, then\ \omega_{n+1} \le \omega_n \le \omega_{n0}$. So, by the definition of the function φ, and the condition (2.6) we can directly obtain (2.10). Now, by (2.1) and (2.10), we get

$$\sum_{l=n_0}^{n-1} q_l \le -\frac{1}{F(\omega_{n0})}\sum_{l=n_0}^{n-1} \Delta\omega_l.$$

Then, for all $n \ge n_0$, we have

$$\sum_{l=n_0}^{n-1} q_l \le -\frac{1}{c}(\omega_n - \omega_{no}), where\ F(\omega_{n0}) = c > 0.$$

Hence, for all $n \ge n_0$, we obtain

$$\frac{\omega_n}{c} \le \frac{\omega_{no}}{c} - \sum_{l=n0}^{n-1} q_l = c_2 - \sum_{l=n_0}^{n-1} q_l,\ whre\ c_2 = \frac{\omega_{n0}}{c}.$$

Then,

$$c^{-1}\frac{k(n, x_n, \Delta x_n)}{g(x_n)} - c_2 \le -\sum_{l=n_0}^{n-1} q_l.$$

Hence, for all $n \ge n_0$, we obtain

$$\frac{b}{c}\frac{\Delta x_n}{g\,(x_n)} - \frac{c_2}{r_n} \le -\frac{1}{r_n}\sum_{l=n_0}^{n-1} q_l.$$

Summing the above inequality from n_0 to $n-1$, we have

$$c_3 \sum_{l=n_0}^{n-1} \frac{\Delta x_l}{g\,(x_l)} - c_2 \sum_{l=n_0}^{n-1} \frac{1}{r_l} \le -\sum_{l=n_0}^{n-1} \left(\frac{1}{r_l}\sum_{m=n_0}^{l-1} q_m\right), \text{where} c_3 = \frac{b}{c}. \tag{2.11}$$

Define $\delta(t) = x_l + (t-l)\Delta x_l, t \in [l, l+1]$. Then we have one of the following two cases:

Case (1): If $\Delta x_l \geq 0$, then $x_l \leq \delta(t) \leq x_{l+1}$. Thus, in view of the definition of the function g, we get

$$\frac{\Delta x_l}{g\,(x_l)} \geq \frac{\delta'(t)}{g(\delta(t))} \geq \frac{\Delta x_l}{g\,(x_{l+1})}\,. \qquad (2.12)$$

Case (2) : If $\Delta x_l \leq 0,$ then$x_{l+1} \leq \delta\,(t) \leq x_l$. So we can directly obtain (2.12).

Now, by (2.11) and (2.12), we get

$$c_3 \int_{n_0}^{n} \left(\frac{1}{g}\right)(\delta(t))d(\delta(t)) \leq \; c_2 \sum_{l=n_0}^{n-1} \frac{1}{r_l} - \sum_{l=n_0}^{n-1} (\frac{1}{r_l} \sum_{m=n_0}^{l-1} q_m).$$

Then, for all $n \geq n_0$, we obtain

$$c_3 \int_{\delta(n_0)}^{\delta(n)} \frac{du}{g(u)} \leq c_2 \sum_{l=n_0}^{n-1} \frac{1}{r_l} - \sum_{l=n_0}^{n-1} (\frac{1}{r_l} \sum_{m=n_0}^{l-1} q_m),$$

Which implies that

$$\int_{\delta(n_0)}^{\delta(n)} \frac{du}{g(u)} \to -\infty \; as \; n \to \infty\,.$$

Now, if $\delta(n) \geq \delta(n_0)$ for large n, then$\int_{\delta(n_0}^{\delta(n)} \frac{du}{g(u)} \geq 0,$

Which a contradiction. Hence, for large $n, \delta(n) \leq \delta(n_0), so$

$$-\int_{\delta(n)}^{\delta(n_0)} \frac{du}{g(u)} \geq -\int_{0}^{\delta(n_0)} \frac{du}{g(u)} > -\infty,$$

Which is again a contradiction. This completes the proof of Theorem 2.2.

Example 2.2 : Consider the difference equation

$$\Delta\left(\left(\frac{1}{n^2} + x_n^2 - 6x_n\Delta x_n + 9(\Delta x_n)^2\right) e^{\Delta x_n} \Delta x_n\right) + (2 + 3(-1)^n)\varphi(u, v) = 0, n \geq 1. \; (2.13)$$

Here, $k(n, x_n, \Delta x_n) = \left(\frac{1}{n^2} + x_n^2 - 6x_n\Delta x_n + 9(\Delta x_n)^2\right) e^{\Delta xn} \, \Delta x_n \geq \frac{1}{n^2} \Delta x_n,$

then $\left(b = 1, \text{and} r_n = \frac{1}{n^2}\right), q_n = 2 + 3(-1)^n, \text{and } \varphi\,(u, v) = u(1 + e^{\frac{-v}{u}}),$

where $u = g(x_{n+1}) = x_{n+1}^5, \text{and } v = \left(\frac{1}{(n+1)^2} + x_{n+1}^2 - 6x_{n+1}\Delta x_{n+1} + 9(\Delta x_{n+1})^2\right) e^{\Delta x_{n+1}}\Delta x_{n+1}.$

All conditions of Theorem 2.2 are satisfied, and hence, all solutions of equation (2.13) are oscillatory.

Note that the Results of E. M.Elabbasy and Sh. R. Elzeiny [5] cannot be applied to (2.13).

In the following, we state and prove some lemmas which will be needed later on.

Lemma 2.1: Assume that there exist positive integers $N_0, N, N \geq N_0$ such that (1.3) holds.

Then there exist an integer $N_1 \geq N$ such that

$$\sum_{i=N_1}^{n} q_i \geq 0 \forall\, n \geq N_1. \tag{2.14}$$

The proof of the above Lemma can be found in [7,Lemma 2.1].

Lemma 2.2: Assume that (1.3) holds, $k(n, x, y) \geq byr_n \forall y \in \mathbb{R}$ and for some constant $b > 0$, and $\varphi(u, v) = u$ in equation (E). Furthemore, suppose that

$$\sum_{n=0}^{\infty}\left(\frac{1}{r_n}\right) = \infty. \tag{2.15}$$

If $\{x_n\}$ is a non-oscillatory solution of equation (E) such that $x_n > 0$ for all$n \geq N_0$, then there exists an integer $N \geq N_0$ such that $\Delta x_n > 0$ for all $n \geq N$.

Proof: If not, assume first that $\Delta x_n < 0$ for all large n, say $n \geq N \geq N_0$.

Without loss of generality, we may assume that (1.3) holds for $n \geq N$ and $q_N \geq 0$.

Define

$$Q_n = \sum_{l=N}^{n} q_l \text{for } n \geq N \ \text{ and} Q_{N-1} = 0. \tag{2.16}$$

Then, we have,

$$\sum_{l=N}^{n} q_l g\,(x_{l+1}) = \sum_{l=N}^{n} g\,(x_{l+1})\Delta Q_{l-1} = \sum_{l=N}^{n}[\Delta(g(x_{l+1})Q_{l-1}) - Q_l \Delta g(x_{l+1})]$$

$$= g(x_{n+2})Q_n - g(x_{N+1})Q_{N-1} - \sum_{l=N}^{n} Q_l \Delta g\,(x_{l+1})$$

$$= g(x_{n+2})Q_n - \sum_{l=N}^{n}(\big(g(x_{l+2}) - g(x_{l+1})\big)Q_l)$$

$$= g(x_{n+2})Q_n - \sum_{l=N}^{n}(g_1(x_{l+2}, x_{l+1})\Delta x_{l+1} Q_l) \geq 0.$$

From equation (E), therefore

$$\sum_{l=N}^{n} \Delta(k(l, x_l, \Delta x_l)) \leq 0.$$

Hence,

$$k(n+1, x_{n+1}, \Delta x_{n+1}) \leq k(N, x_N, \Delta x_N)$$

$$< \;\; 0,$$

since

$$k(n+1, x_{n+1}, \Delta x_{n+1}) \geq b r_{n+1} \Delta x_{n+1}.$$

Then ,

$$\Delta x_{n+1} \leq \frac{c_4}{r_{n+1}} < 0, \qquad c_4 = \frac{k(N, x_N, \Delta x_N)}{b} < 0. \tag{2.17}$$

Summing (2.17) from N to $n-1$, we obtain

$$x_{n+1} - x_N < c_4 \sum_{l=N+1}^{n} \frac{1}{r_l}$$

Then , we get

$x_{n+1} \to -\infty \; as \; n \to \infty$,which a contradiction.

Next, assume that Δx_n is oscillatory for $n \geq N_1 \geq N \geq N_{0.}$ Then there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ with$\lim_{k\to\infty} \mathrm{n}_k = \infty$ and such that$\Delta x_{n_k} = 0, k = 1,2,3 ....$.

Letting

$$\omega_n = \frac{k(n, x_n, \Delta x_n)}{g(x_n)}, n \geq N_1.$$

Then, for all $n \geq N_1$,we obtain

$$\Delta\omega_n = \frac{\Delta(k(n, x_n, \Delta x_n))}{g(x_{n+1})} - \frac{k(n, x_n, \Delta x_n)\Delta g(x_n)}{g(x_n)g(x_{n+1})}$$

$$= \frac{-q_n\, g(x_{n+1})}{g(x_{n+1})} - \frac{k(n, x_n, \Delta x_n) g_1(x_{n+1}, x_n)(\Delta x_n)^{\delta}}{g(x_n)g(x_{n+1})}$$

$$\leq\ -q_n, n \geq N_1.$$

Then,

$$q_n \leq -\Delta\omega_n, \qquad n \geq N_1.$$

Summing the above inequality fromn_1 to $n_k - 1$, we have

$$\sum_{l=n_1}^{n_{k-1}} q_l \leq -\omega_{n_k} + \omega_{n_1} = 0,$$

Which contradicts (2.14). Hence $\Delta\omega_n > 0$ for all$n \geq N_1$.

Theorem 2.3 : Assume that (1.3) and (2.15) hold, $k(n, x, y) \geq byr_n\ \forall\ y \in\ \mathbb{R}$ and for some constant $b > 0$, and $\varphi(u, v) = u$ in equation (E). Furthermore, assume that there exists $\lambda \geq 1$ such that

$$\lim_{m\to\infty} sup \frac{1}{m^{\lambda}} \sum_{n=n_0}^{m-1} (m-n)^{\lambda}\, q_n = \infty. \qquad (2.18)$$

Then every solution of Equation (E) oscillates.

Proof: suppose to the contrary that $\{x_n\}$ is a non oscillatory solution of (E).

Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive solution of (E), such that $x_n > 0$ for all large *n*. In view of lemma 2.2, we see that, there is some $n_1 \geq n_0$ such that

$$x_n > 0, \qquad \Delta x_n > 0, n \geq n_1.$$

Define the sequence $\{\omega_n\}$ by

$$\omega_n = \frac{k(n,x_n,\Delta x_n)}{g(x_n)}, n \geq n_1, \text{ then } \omega_n > 0 \text{ and } q_n \leq -\Delta\omega_n.$$

Hence,

$$\sum_{n=n_1}^{m-1} (m-n)^\lambda q_n \leq -\sum_{n=n_1}^{m-1} (m-n)^\lambda \, \Delta\omega_n. \tag{2.19}$$

But

$$-\sum_{n=n_1}^{m-1} (m-n)^\lambda \Delta\omega_n = (m-n_1)^\lambda \omega_{n_1} - \sum_{n=n_1}^{m-1} \omega_{n+1}\left[(m-n)^\lambda - (m-n-1)^\lambda\right].$$

By means of the well-known inequality [8]

$x^\beta - y^\beta \geq \beta y^{\beta-1}(x-y)$ for all $x \geq y > 0$ and $\beta \geq 1$,

We have,

$$-\sum_{n=n_1}^{m-1} (m-n)^\lambda \Delta\omega_n \leq (m-n_1)^\lambda \omega_{n_1} - \sum_{n=n_1}^{m-1} \lambda\omega_{n+1}(m-n-1)^{\lambda-1}$$
$$\leq (m-n_1)^\lambda \omega_{n_1}. \tag{2.20}$$

Then by (2.19) and (2.20), we get

$$\sum_{n=n_1}^{m-1} (m-n)^\lambda q_n \leq (m-n_1)^\lambda \omega_{n_1},$$

Which implies that

$$\frac{1}{m^\lambda}\sum_{n=n_1}^{m-1} (m-n)^\lambda q_n \leq \left(\frac{m-n_1}{m}\right)^\lambda \omega_{n_1}.$$

Hence,

$$\lim_{m\to\infty} sup \frac{1}{m^{\lambda}} \sum_{n=n_1}^{m-1} (m-n)^{\lambda} q_n < \infty,$$

Which is contradiction to (2.18). The proof is completed.

Theorem 2.4 : Assume that (1.3) and (2.15) hold, $k(n,x,y) \geq byr_n \ \forall\ y \in \mathbb{R}$ and for some constant $b > 0$, and $\varphi(u,v) = u$ in equation (E). Furthermore, assume that there exists a positive sequence $\{p_n\}_{n=0}^{\infty}$ such that $\Delta P_n \leq 0$ for all$n \geq n_0 > 0$, and

$$\sum_{n=n_0}^{\infty} \rho_{n+1} q_n = \infty, \text{ for some } n_0 > 0, \qquad (2.21)$$

Then every solution of equation (E) oscillates.

Proof: Suppose to the contrary that $\{x_n\}$ is a nonoscillatory solution of (E).

Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive solution of (E) such that $x_n > 0$ for all $n \geq n_0 > 0$. Then, $g(x_{n+1}) > 0$ for all $n \geq n_0 > 0$.

Then, from Lemma 2.2, there exists an integer $n_1 \geq n_0$, sufficiently large, so that

$$\Delta x_n > 0 \text{ forall } n \geq n_1.$$

Now,

$$\Delta\left(\frac{\rho_n k(n,x_n,\Delta x_n)}{g(x_n)}\right) = \frac{\rho_{n+1}}{g(x_{n+1})}\Delta\big(k(n,x_n,\Delta x_n)\big) + k(n,x_n,\Delta x_n)\Delta\left(\frac{\rho_n}{g(x_n)}\right)$$

$$\leq\ -\rho_{n+1}q_n + \frac{k(n,x_n,\Delta x_n)\Delta\rho_n}{g(x_{n+1})} - \frac{\rho_n k(n,x_n,\Delta x_n)\Delta(g(x_n))}{g(x_n)g(x_{n+1})}$$

$$\leq\ -\rho_{n+1}q_n + \frac{k(n,x_n,\Delta x_n)\Delta\rho_n}{g(x_{n+1})} - \frac{\rho_n k(n,x_n,\Delta x_n)\Delta x_n}{g(x_n)g(x_{n+1})}$$

$$\leq\ -\rho_{n+1}q_n, \quad \text{forall } n \geq n_1.$$

Hence ,

$$\rho_{n+1}q_n \leq \Delta\left(\frac{\rho_n k(n,x_n,\Delta x_n)}{g(x_n)}\right).$$

Summing the above inequality from n_1 to $n-1$, we obtain

$$\sum_{m=n_1}^{n-1} \rho_{m+1} q_m \leq \frac{\rho_{n_1} k\left(n_1, x_{n_1}, \Delta x_{n_1}\right)}{g\left(x_{n_1}\right)} - \frac{\rho_n k(n, x_n, \Delta x_n)}{g(x_n)}$$

$$\leq \frac{\rho_{n_1} k\left(n_1, x_{n_1}, \Delta x_{n_1}\right)}{g\left(x_{n_1}\right)},$$

which is contrary to (2.21). The proof is completed.

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