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The Convergence of Polynomial Interpolation and Runge Phenomenon

Souad Abumaryam

e-mail: s.ahmed @su.edu.ly

Mathematics Department, Faculty of Science, Sirte University

Abstract

One of the main questions is whether or not a sequence of polynomials $p_n(x)$ that interpolate a continuous function f at $n+1$ equally spaced points tends to f in the sup-norm? The answer is "no" in some cases. The main fact is that interpolant polynomials $p_n(x)$ of a function f converge at a rate determined by the smoothness of f: the $p_n(x)$ converge rapidly to the function f if it is k-times differentiable and converges exponentially if f is analytic. The polynomial interpolation depends on n but it also depends on the way in which the points are distributed. We determine conditions on the function f to ensure the convergence of the polynomials $p_n(x)$ to the function f, as the continuity of the function is not enough. The question for analytic functions is answered using potential theory. Convergence and divergence rate of interpolants of analytic functions on the interval are investigated. We also study a generalized Runge phenomenon and find out how the location of the points and poles affect the convergence.

Keywords: *Polynomial interpolation, Lagrange polynomial, Chebyshev polynomial, Chebyshev points.*

1. Introduction

Interpolation theory is one of the most important tools of numerical analysis. It is used in approximation theory, numerical differentiation, numerical integration and solving differential equations. The interpolating polynomial is the polynomial that passes through the given data values with degree less than the number of points. Polynomials are the simplest form of interpolation to work with, this is due to the ease of their differentiation and integration. The interpolating polynomial is the polynomial that passes through $n+1$ data points and whose degree is at most n may give a good approximation for small n. However, interpolation at equally spaced points does not always yield a good approximation, for example, due to the Runge phenomenon. Yet, interpolation at Chebyshev points leads to better results. Even better results can be obtained through interpolation in term of the Chebyshev polynomial. There are several different types of interpolation with certain merit and demerit. For example, the Lagrange polynomials interpolation formula is of a theoretical interest but in practice is not satisfactory. Instead, there are several attractive alternatives: the modified Lagrange and the barycentric formula. The barycentric formula is known to be a good method because of attractive features e.g. stability, as shown by Higham [12].

Polynomial interpolation is the dominant for approximation and has some clear advantages. For instance, any continuous function on a given interval $[a, b]$ can be approximated by polynomials ($Weierstrass\ theorem$). But there are some disadvantages, as a high polynomial degree is generally needed for accuracy, which in some cases leads to divergence. In fact, polynomial interpolants at evenly spaced points need to converge uniformly for continuous function as $n \to \infty$ even if the function is analytic [14], (see also a simple example.2) with the Runge phenomenon). If we are able to choose the points of interpolation, then remedy is to interpolate using points that are clustered at the end of intervals, such as $Chebyshev\ points$ [10]. If we cannot choose the points, then we have to use another approach.

2. The Chebyshev polynomial

The Chebyshev polynomial of the first kind of degree n is defined as [4, 9]:

$$T_n(x) = cos(n\ cos^{-1} x) = cos\, n\theta, \qquad (0.1)$$

where $x = cos\theta, -1 \le x \le 1,\ 0 \le \theta \le \pi$ and n is nonnegative integer.

The Chebyshev polynomials $T_n(x)$ satisfy $|T_n(x)| \le 1$. This follows from the bound $|cos\,\theta| \le 1$, thus

$$|T_{n+1}(x) - T_{n-1}(x)| \le 2$$

The Chebyshev polynomial $T_n(x)$ of degree $n > 1$ has n zeros on the interval $[-1, 1]$. The zeros x_j are given by:

$$x_j = \frac{cos(2j-1)\pi}{2n}, \quad j = 1, \dots, n$$

Moreover, the extrema (i.e. points $\tilde{x}_j$ such that $T_n(\tilde{x}_j) =$ $(-1)^j$ are given by

$$\tilde{x}_j = \frac{cos\, j\pi}{n}, \quad j = 1, \dots, n$$

All roots are real and lie in the interval $[-1, 1]$. The extrema are preferable for interpolation in practical use because they include the boundary points.

Theorem.1 [9,15] : *A function $f\ (x)$ on [−1, 1] that satisfies the Lipschitz continuity condition can be expanded as a Chebyshev series*

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\ T_k(x), \qquad (1.1)$$

which converges uniformly and absolutely on $[-1,1]$, *where*

$$a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}} \qquad (2.1)$$

The Chebyshev polynomials have interesting properties that make them a very attractive tool to minimize the maximum error in uniform approximation.

3. Barycentric Polynomial Interpolation

Let $\{x_j\}_{j=0}^{n}$ $n+1$ distinct given points with associated values $\{f_j\}_{j=0}^{n}$, which may or may not be values of a function $f(x)$: $f_j = f(x_j)$ $j = 0, \dots, n$. Then there is a unique polynomial $p_n(x)$ of a degree $\leq n$ such that

$$p_n(x_j) = f(x_j), \qquad j = 1, \dots, n$$

The polynomial $p_n(x)$ is called the interpolating polynomial [10, 15]

An interpolating polynomial can be constructed easily using the Lagrange formula

$$p_n(x) = \sum_{j=0}^{n} f(x_j) L_J(x), \qquad (3.2)$$

where the Lagrange polynomial basis L_J is

$$L_J(x) = \prod_{\substack{k=0 \\ k\neq j}}^{n} \frac{x - x_k}{x_j - x_k}, \quad k = 0, \dots, n \qquad (4.2)$$

L_J is the unique polynomial of degree n that have the property

$$L_J(x_k) = \delta_{jk} = \begin{cases} 1 & if\ \ j = k \\ 0 & if\ \ j \neq k \end{cases}$$

where δ_{jk} is the Kronecker delta.

The Lagrange interpolation formula is useful for theoretical interest but not appropriate in practice [10].

The Lagrange formula (3.2) can be rewritten in a different and more attractive way. Let us define:

$$L(x) = \prod_{k=0}^{n} (x - x_k).$$

Its derivative is

$$L^{'}(x) = \sum_{j=0}^{n} \prod_{\substack{k=0 \\ k \neq j}}^{n} (x - x_k).$$

Thus, when $L^{'}(x)$ is evaluated at an interpolation point x_j, there will be only one term not equal to zero and thus $L^{'}(x)$ is equal to

$$L^{'}(x_j) = \prod_{\substack{k=0 \\ k \neq j}}^{n} (x_j - x_k).$$

Hence, from (4.2), L_J becomes

$$L_J(x) = \frac{L(x)}{L^{'}(x_j)(x - x_j)}. \qquad (5.2)$$

If the weight is defined by

$$w_j = \frac{1}{\prod_{k=0}^{n} (x_j - x_k)} = \frac{1}{L^{'}(x_j)}$$

then (4.2) can be written as

$$L_J(x) = L(x) \frac{w_j}{(x - x_j)}.$$

Thus, (3.2) can be expressed as

$$p_n(x) = \sum_{j=0}^{n} w_j f_j \prod_{\substack{k=0 \\ k \neq j}}^{n} (x - x_k).$$

Therefore, the improved Lagrange formula is defined as:

$$p_n(x) = L(x) \sum_{j=0}^{n} f_j \frac{w_j}{(x - x_j)} \qquad (6.2)$$

Note that the denominator of $L_J(x)$ can be computed in advance because it is constant. Here, the coefficient of x^n is

$$\sum_{j=0}^{n} f_j \prod_{k=0}^{n} (x_j - x_k) = \sum_{j=0}^{n} \frac{f_j}{L^{'}(x_j)}.$$

The formula (6.2) is called the modified Lagrange formula or the first form of the barycentric interpolation formula and enjoys some advantages over the Lagrange formula [10].

We now arrive at the formula that leads to a better numerical computation. The barycentric formula can be obtained from the modified Lagrange formula (6.2) [10]. The interpolation of $f_j = 1$ for all j is the unique polynomial $p_n(x) = 1$, which has zero degree. Hence, (6.2) implies that

$$1 = \sum_{j=0}^{n} L_J(x) = L(x) \sum_{j=0}^{n} \frac{w_j}{(x - x_j)} \qquad (7.2)$$

So

$$L(x) = \frac{1}{\sum_{j=0}^{n} \frac{w_j}{(x - x_j)}}.$$

By dividing (3.2) by (7.2) we get

$$p_n(x) = \frac{\sum_{j=0}^{n} f(x_j) L_J(x)}{\sum_{j=0}^{n} L_J(x)}$$

Substituting (6.2) into the above and cancelling $L(x)$ leads to

$$p_n(x) = \frac{\sum_{j=0}^{n} f(x_j) \frac{w_j}{(x - x_j)}}{\sum_{j=0}^{n} \frac{w_j}{(x - x_j)}} \qquad (8.2)$$

which is known as the barycentric formula. It is an a polynomial if the weights w_j are nonzero and defined in a such a way that

$$\sum_{j=0}^{n} L_j = L(x) \sum_{j=0}^{n} \frac{w_j}{(x - x_j)} = 1.$$

It was mentioned by Taylor in 1945 [1] for equally spaced points. Later, the formula was reconsidered by Salzer in [5] in 1972 for Chebyshev points. Furthermore, the formula was discussed in 2004 by Berrut and Trefethen [10]. Since then, the formula has become widely used for interpolation and opened up a wide field of research.

In some cases the weight w_j can be computed analytically, for example in the case of equidistant interpolation points $x_j = a + jh$, where $h = \frac{b-a}{n}$ by using

$$w_j = \frac{1}{\prod_{k=0}^{n} (x_j - x_k)} = \frac{1}{L'(x_j)}$$

Then,

$$\prod_{k=0}^{n}(x_j - x_k) = \left[\prod_{k=0}^{j-1}(x_j - x_k)\right]\left[\prod_{k=j+1}^{n}(x_j - x_k)\right]$$

$$\prod_{k=0}^{n}(x_j - x_k) = \left[h^j\prod_{k=0}^{j-1}(j-k)\right]\left[h^{n-j}\prod_{k=j+1}^{n}(j-k)\right]$$

$$= \left[h^j\prod_{k=0}^{j}k\right]\left[(-1)^{n-j}\prod_{k=1}^{n-j}k\right]$$

$$= (-1)^{n-j}h^n\,(j!)\,\big((n-j)!\big).$$

After some cancellation of a common factor, we have

$$w_j = (-1)^j\binom{n}{j}$$

Chebyshev points are obtained by projecting equidistant points on the unit circle onto the interval $[-1,1]$:

$$x_j = \frac{cos(2j-1)\pi}{2n}\ , \quad j = 1,\dots,n$$

These points are clustered at the ends of the interval. Here, after cancelling common factors that are independent of j, the weights become

$$w_j = (-1)^j\ \frac{sin(2j-1)\pi}{2n}\ , \quad j = 1,\dots,n,$$

and the interpolation formula becomes:

$$p_n(x) = \frac{\sum_{j=0}^{n} f(x_j)\dfrac{(-1)^j\frac{sin(2j-1)\pi}{2n}}{(x-x_j)}}{\sum_{j=0}^{n}\dfrac{(-1)^j\frac{sin(2j-1)\pi}{2n}}{(x-x_j)}}.$$

Also, if the points are the Chebyshev extrema points

$$x_j = \frac{cos\, j\pi}{n}, \quad j = 1,\dots,n$$

then the weights become $w_j = (-1)^j\delta_j$ [10], where

$$\delta_j = \begin{cases}\frac{1}{2} & if\ \ j=0 \quad j=n\\ 1 & otherwise\end{cases}$$

For the interval $[a,b]$ the weights will be multiplied by $2^n(b-a)^n$.

Using the data f_j at $n+1$ Chebyshev points x_j, the interpolant polynomials simplify to a formula by Salzer [5]

$$p_n(x) = \frac{\sum_{j=0}^{n} f(x_j)\frac{(-1)^j}{(x-x_j)}}{\sum_{j=0}^{n}\frac{(-1)^j}{(x-x_j)}}.$$

4. Main Result: Runge Phenomenon

In many cases a high-order polynomial interpolation leads to a good approximation. However, as the following examples demonstrate, this is not true for all continuous functions on a finite interval $[a, b]$.

Example.1: Bernstein proved that for $f(x) = |x|$, the interpolating polynomial $p_n(x)$ converges only at the points $x = -1, 0$ and 1, where the points 1 and -1 are interpolation points satisfying $p_n(1) = p_n(-1) = 1$. For the point 0, Natanson in [2] proved that $lim_{n\to\infty} p_n(0) = 0$ when n is even but not for odd n.

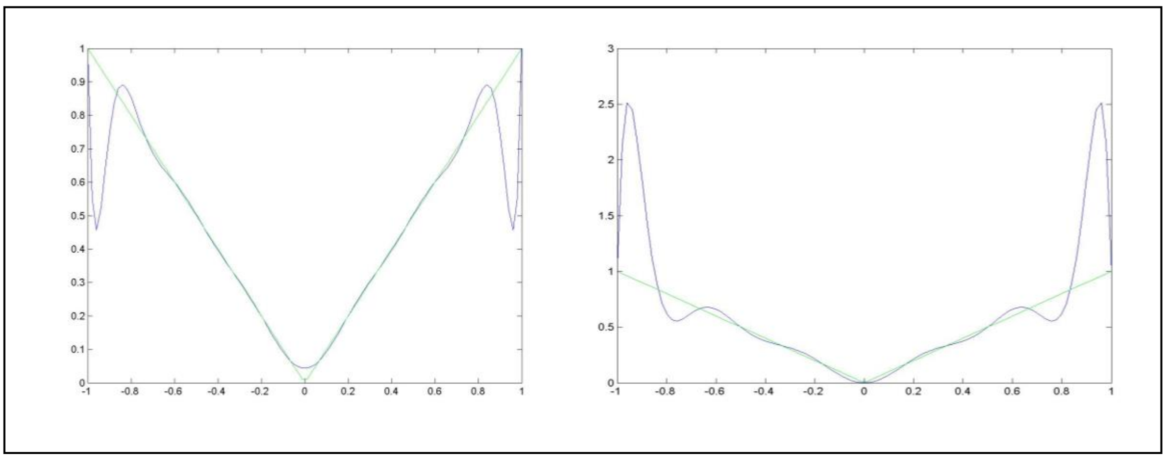

Figure 1. Polynomial interpolations of f(x) = |x| on $[-1, 1]$ at equally spaced points n = 16 (left) and n = 13 (right).

Moreover, if the function is infinitely differentiable but is not bounded or analytic, then the error may not be small as the number of interpolation points increases, unless the function is analytic in a larger complex region where its shape depends on the interpolation points.**Runge's phenomenon** shows that using equally spaced points can lead to a loss of accuracy in the interpolating polynomials.

Example.2: If the function $f(x) = \frac{1}{1+x^2}$ on $[-5,5]$ is approximated at $x_j = -5 + hj$, where $h = \frac{10}{n}$, then the error becomes worse as the number of points increases (see Figure2).

This causes large oscillations near to the end of the interval, although convergence takes place in a smaller interval $[-a, a]$ *with* $a \leq 3.63$. However, increasing the degree leads to good convergence in the middle of the interval.

The main reason for the divergence of the interpolating polynomial at equally spaced points is that the function is analytic for all x, but has poles at $x = \pm i$.

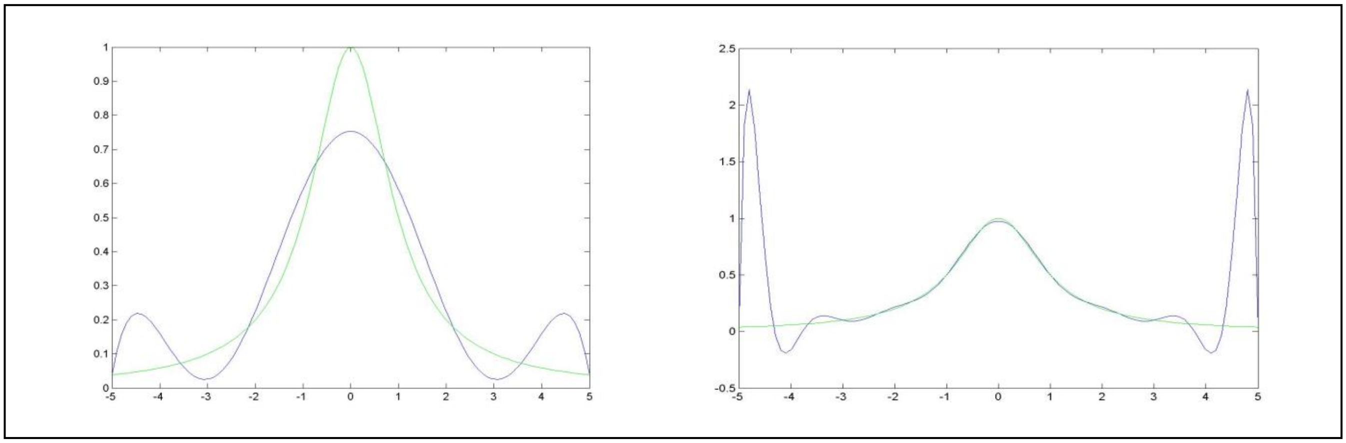

Figure.2: Polynomial interpolation of $f(x) = \frac{1}{1+x^2}$ on $[-1, 1]$ at equally spaced for $n = 8$ (left) and $n = 16$ (right).

Berrut and Trefethen in [10] outlined the way to avoid this problem by using points clustered at the end of the interval, such as Chebyshev points. This is a notable improvement, with no oscillation at the end of the interval. However, there exist continuous functions for which interpolation at Chebyshev zeros does not converge. Such a function does not satisfy the Lipschitz condition.

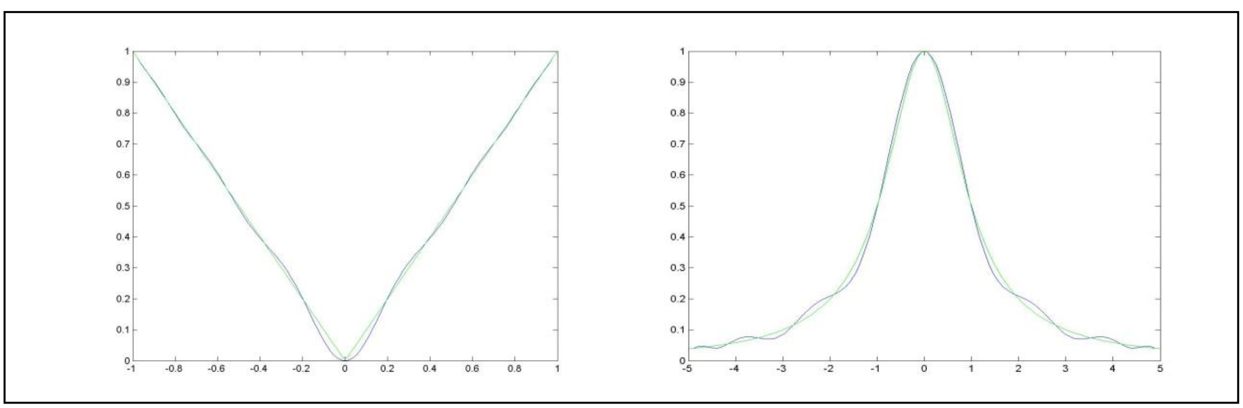

Figure.3. Polynomial interpolation on $[-1, 1]$ of $f(x) = |x|$ at Chebyshev points for $n = 16$ (left) and $f(x) = \frac{1}{1+x^2}$ for $n = 16$ (right).

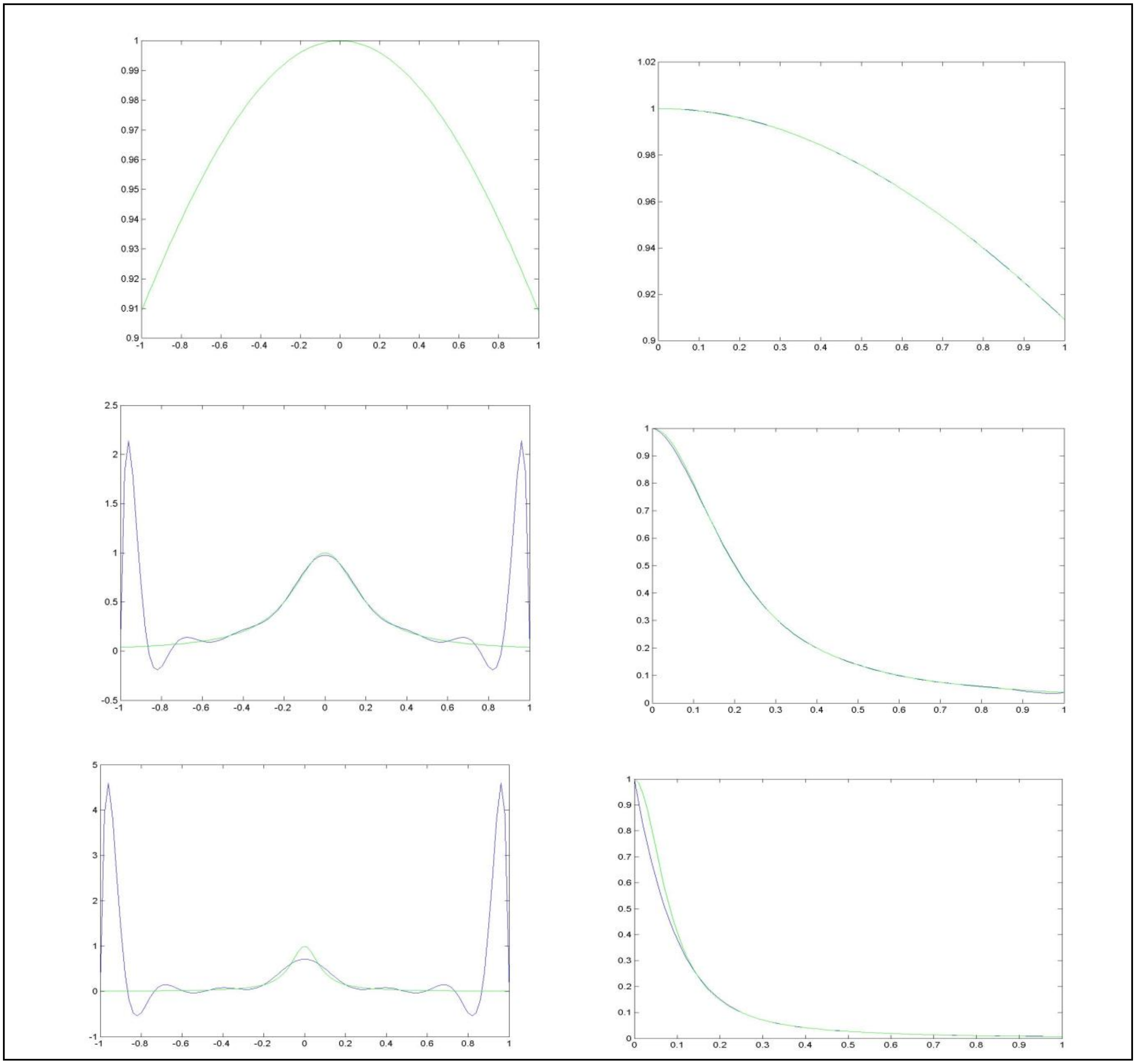

Figure4: Polynomial interpolation of functions with different singularities. The top left panel is $f(x) = \frac{1}{1+0.1\,x^2}$ on $[-1,1]$ with $n = 16$ and the top right panel on $[0,1]$ with $n = 8$. In the middle left panel we have $f(x) = \frac{1}{1+25\,x^2}$ on $[-1,1]$ with $n = 16$ and the middle right panel on $[0,1]$ with $n = 8$. In the bottom left panel is $f(x) = \frac{1}{1+144\,x^2}$ on $[-1,1]$ with $n = 16$ and the bottom right panel on $[0,1]$ with $n = 16$ and the bottom right panel on $[0,1]$ with $n = 8$. Observation: There is no Runge phenomenon on the half-interval.

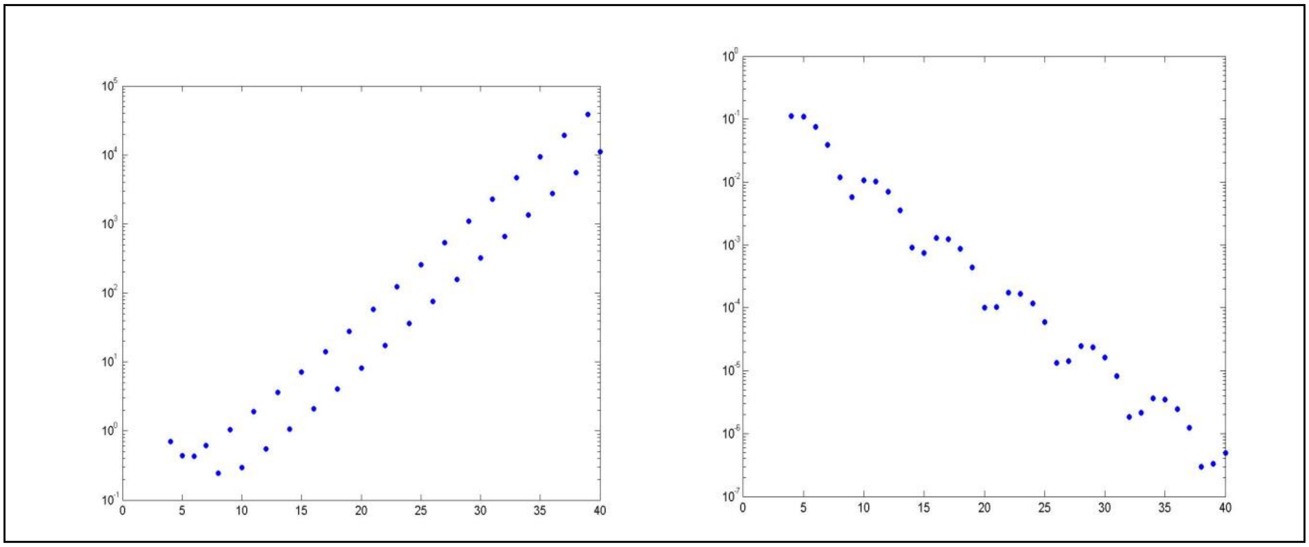

Figure5. Convergence of the error in the sub-norm (∞-norm) of the polynomial interpolation. The left panel is for the function $f(x) = \frac{1}{1+\ x^2}$ at equally spaced points on the interval $[-5,5]$ and the right for $[0,5]$. The horezental axes refers to the number of points while the vertical axes refers to

$\|f(x) - p_n(x)\|_\infty$.

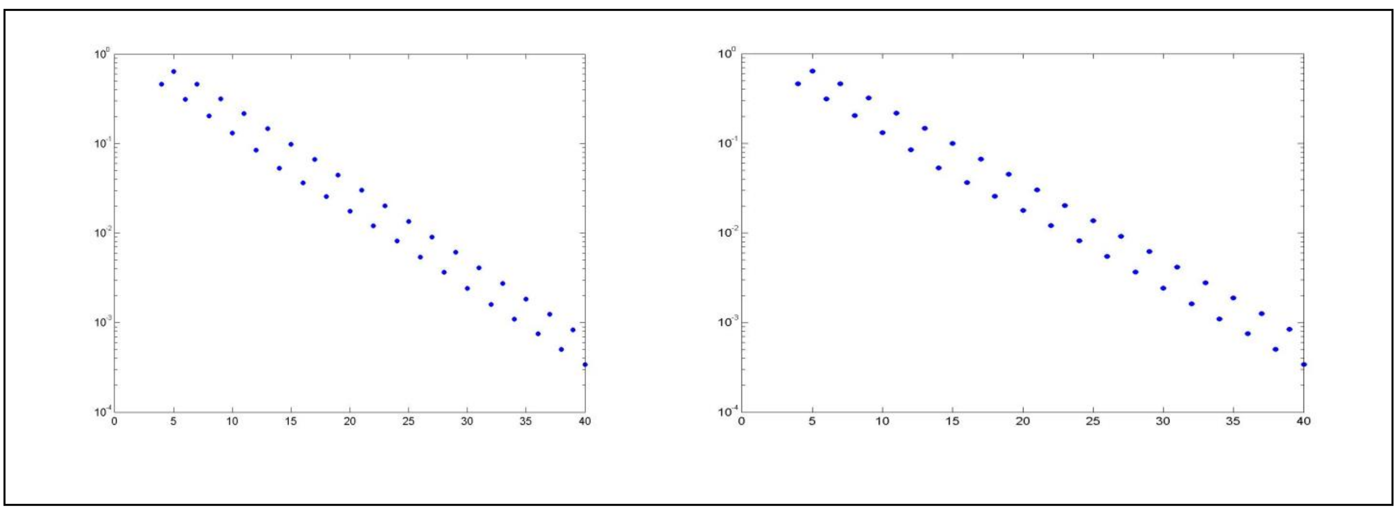

Figure6. Convergence of the error in the sub-norm (∞-norm) of the polynomial interpolation. The left panel is for the function $f(x) = \frac{1}{1+\ x^2}$ at Chebyshev points on the interval $[-5,5]$ and the right for $[0,5]$. The horezental axes refers to the number of points while the vertical axes refers to $\|f(x) - p_n(x)\|_\infty$.

We observe from Figure2 that the Gaussian function $e^{-\tau x^2}$ and $\arctan(\tau x)$ with equally spaced points are susceptible to the Runge phenomenon, depending on τ and n.

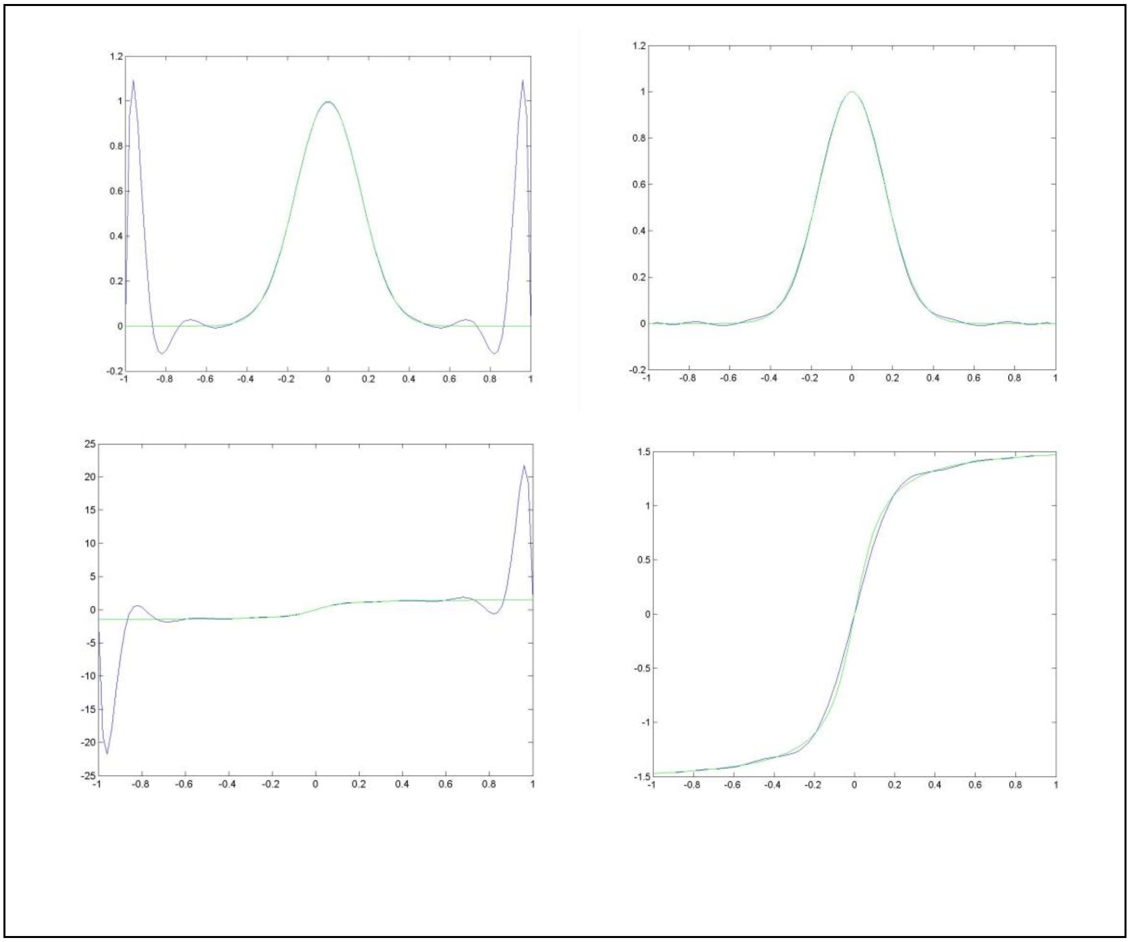

Figure7: Polynomial interpolation for the functions $f_1(x) = e^{-20x^2}$ and $f_2(x) = \arctan(20x)$. The top left panel shows the interpolation $f_1(x) = e^{-20x^2}$ using equally spaced points and the top right panel using Chebyshev points, both with $n = 16$. The bottom left panel shows the interpolation for $f_2(x) = \arctan(20x)$ using equally spaced points and the bottom right panel using Chebyshev points, both with $n = 16$. Observation: There is an oscillation (Runge phenomenon) at the end of interval for the polynomial interpolant of the functions $f_1(x)$ and $f_2(x)$ at equally spaced points.

If we interpolate the function on half of the interval or any subinterval, say [0,5] or [0,1], at equally spaced points with the same total number of points, we will get a better result. The interpolant converges faster to the function on [0,5] than on $[-5,5]$ (see Figure 8).

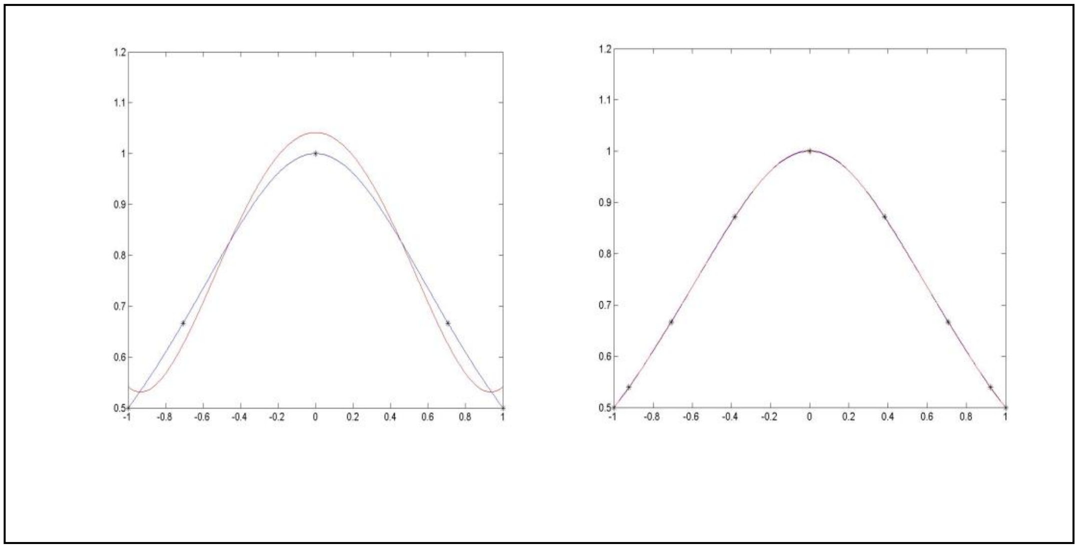

Figure 8. Polynomial interpolation of the function $f(x) = \frac{1}{1+ x^2}$ in terms of Chebyshev polynomials for $n = 4$ (left) and $n = 8$ (right). Observation: There is no oscillation at the end of interval (Runge phenomenon).

We now consider the case when the function f(x) extends to a function f(z) of the complex plane which is analytic in a simple closed contour C that contains the interval $[a, b]$. The complex error function is given by a contour integral:

Theorem.2 [3] : *Assume that f is a function that extends to an analytic function in a domain Ω that contains the interval $[-1,1]$. Let $C \subset \Omega$ be a simple closed contour in the complex plane and let $x_j \in C$, where f is an analytic function on and inside C. Then*

$$f(x) - p_n(x) = \frac{1}{2\pi i}\int_C \frac{\phi_n(x) f(z)}{\phi_n(z)(z-x)} dz, \qquad x \in [-1,1], \qquad (9.3)$$

Where

$$p_n(x) = \frac{1}{2\pi i}\int_C \frac{f(z)\left(\phi_n(z) - \phi_n(x)\right)}{\phi_n(z)(z-x)}, \quad \text{where} \quad \phi_n(x) = \prod_{k=0}^{n}(x - x_k).$$

From (4.2), the polynomial interpolant can be written as

$$p_n(x) = \sum_{j=0}^{n} \frac{\phi_n(x) f(x_j)}{\phi_n'(x_j)(x - x_j)} \qquad (10.3)$$

Therefore (9.3) has a simple pole at $z = x$ *with residue* $\frac{f(z)}{\phi_n(z)}$ *and simple poles* $z = x_j$ *with residue* $\frac{f(x_j)}{L'(x_j)\,(x_j - x)}$.

By subtracting (9.3) from

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - x)}\, dz$$

Then, we have (10.3) *which is a polynomial.*

Taking the absolute values, we get estimation:

$$|f(x) - p_n(x)| \leq \frac{1}{2\pi i} \max_{x\in[-1,1]} |\phi_n(x)| \int_C \frac{|f(z)|}{|\phi_n(z)(z - x)|} |dz|$$

$$\leq \text{Const}\, \frac{|\phi_n(x)| \max\limits_{z\in C} |f(z)|}{\min\limits_{x\in[-1,1]} |\phi_n(z)||z - x|}$$

We can obtain the following estimation:

Theorem.3 : *Let* $f(x)$ *be a rational function on* $[a, b]$. *Then the interpolant* p_n *converges to* $f(x)$ *if* $|b - a| < R$, *where* R *is the shortest distance from* $[a, b]$ *to the singularities of the function.*

Proof*:* We start with the Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} (z - x)^n$$

which converges for an analytic function f inside the circle $|z - x| < R > 0$, where R is the shortest distance from $[a, b]$ to the singularities of the function.

By using the Cauchy formula, we have

$$f^{(n)}(x) = \frac{n!}{2\pi}\int_c \frac{f(z)dz}{(z-x)^{n+1}}$$

where C is a closed contour. Therefore, the estimation for the above is given by

$$\left|f^{(n)}(x)\right| \le \frac{n!}{2\pi}\int_c \frac{|f(z)||dz|}{|(z-x)^{n+1}|} \le \frac{Mn!}{R^n}$$

Then,

$$|f(x) - p_n(x)| \le \frac{Mn!\,(b-a)^{n+1}}{R^n n!}$$

Therefore, if $|b-a| < R$ then the interpolant p_n converges to f.

In the case of a meromorphic function, the error can be represented in terms of contour integral in the complex plane by using the residue theorem:

Theorem.4 : *Assume that $f(z)$ is a function that is analytic except for a finite number of poles. Then the p_n converge to $f(x)$ if*

$$\max_{a\le x\le b}\left|\frac{\phi_n(x)}{\phi_n(x_j)}\right| \to 0 \quad \text{as} \quad n\to\infty, \quad \text{where} \quad \phi_n(x) = \prod_{k=0}^{n}(x-x_k).$$

Proof: Let $C(\rho)$ and $C(\rho^*)$be two contours, where the contour $C(\rho^*)$ encloses a simple pole of the function and the contour $C(\rho)$ encloses the interval and the interpolation points but does not enclose any singularities of the function. Let $C(\rho')$ enclose $C(\rho)$ and $C(\rho^*)$. Then, the formula (9.3) is equivalent to

$$\frac{1}{2\pi i}\int_{C(\rho)}\frac{\phi_n(x)f(z)}{\phi_n(z)(z-x)}dz = \frac{1}{2\pi i}\int_{C(\rho')}\frac{\phi_n(x)f(z)}{\phi_n(z)(z-x)}dz - \frac{1}{2\pi i}\int_{C(\rho^*)}\frac{\phi_n(x)f(z)}{\phi_n(z)(z-x)}dz$$

Using the residue theorem we have

$$f(x) - p_n(x) = \frac{1}{2\pi i}\int_{C(\rho')}\frac{\phi_n(x)f(z)}{\phi_n(z)(z-x)}dz - \sum_{j=0}^{m}\operatorname{Res}\left[\frac{\phi_n(x)f(z)}{\phi_n(z)(z-x)}\right] \qquad (11.3)$$

where Res f *is the residue of* f *at* x_j.

Since the function $f(z)$ is a rational function. Then, the first integrand of (11.3) would be zero for sufficiently large n as its radius tends to infinity, which is given by the residue at infinity. Leaving the integral around the poles, which is equal to sum of the residues at the poles. So

$$f(x)-\ p_n(x)=-\sum_{j=0}^{m}\left[\frac{\phi_n(x)}{\phi_n(z)(z-x)}\right]\mathrm{Res}[f(z)],\quad \text{for all}\quad z=x_j,$$

and $z=x_j$ *is a pole of* $f(z)$. *Then, the integral around the poles is computed by the residue of the function at each pole:*

$$\max_{a\le x\le b}|f(x)-\ p_n(x)|\le\sum_{j=0}^{n}\max_{a\le x\le b}\left|\frac{\phi_n(x)}{\phi_n(x_j)(x_j-x)}\right||\mathrm{Res}[f(z)]|,$$

$$\le\ \mathrm{Const}\ \max_{a\le x\le b}\left|\frac{\phi_n(x)}{\phi_n(x_j)}\right|$$

Therefore, the error converges uniformly to zero only if

$$\max_{a\le x\le b}\left|\frac{\phi_n(x)}{\phi_n(x_j)}\right|\to 0,\qquad \text{as}\quad n\to\infty.$$

More generally, the error bound can be estimated:

Lemma.1 [3]: *Assume that* f *is analytic in a region* Ω *bounded by a simple closed contour* C *that encloses the interpolation points* $\{x_j\}_{j=0}^{n}$. *Then*

$$|f(x)-\ p_n(x)|\le\frac{L(C)\max|f(z)|}{\min\limits_{x\in[-1,1]}|z-x|}\quad e^{n\max\kappa(z)}\quad z\in C$$

where $\kappa(z)=\frac{1}{n}\max_{x\in[-1,1]}\log\left|\frac{\phi_n(x)}{\phi_n(z)}\right|$ and $L(C)$ is the arc length.

The convergence of the function is based on the function $\left|\frac{\phi_n(x)}{\phi_n(z)}\right|$ and the pole location. The location of the points plays a crucial part in polynomial interpolation and its convergence or divergence.

The region of convergence can be explained by potential theory.

5. Potential theory and Runge's phenomenon

Convergence or divergence of a polynomial interpolant depends on the domain of analyticity of the function being interpolated. To explain Runge's phenomenon, we start with a result from potential theory that is related to Lemma.1 above:

Let $\phi_n(x)$ be a polynomial such that $\phi_n(x) = \prod_{k=0}^{n}(x - x_k)$, where it can be extended to $z \in \mathbb{C}$

$$\phi_n(z) = \prod_{k=0}^{n}(z - x_k)$$

Then taking the logarithm of both sides, we have

$$\log|\phi_n(z)| = \sum_{k=0}^{n} \log|z - x_k|$$

Now, let us define the discrete potential [8]

$$\delta_n(z) = \frac{1}{n+1}\sum_{k=0}^{n} \log|z - x_k|.$$

$\delta_n(z)$ is the potential at z, which depends on the charge $\frac{1}{n+1}\sum_{k=0}^{n} \log|z - z_k|$ at each z_k.

There is a correspondence between $\phi_n(z)$ and $\delta_n(z)$:

$$|\phi_n(z)| = e^{n\delta_n(z)}$$

A small variation in $\delta_n(z)$ leads to exponentially larger variation in $\phi_n(z)$ for large n. From this, the Runge phenomenon is related to potential theory. If the points are equally spaced, then the above sum can be defined as a Riemann sum for the integral.

*L*et assume the distribution of the interpolation points x_j in $[-1,1]$ is given by a density function $\mu(x)$ with

$$\int_b^a \mu(x)\, dx = \int_{-1}^{1} \mu(x)\, dx = 1$$

and define the associated potential function

$$\delta(z) = \int_{-1}^{1} \mu(x) \log|z - x|\, dx$$

The normalized point measure μ is given by

$$\mu_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} \sigma(x - x_j),$$

where σ is the Dirac delta functions of strength1, and μ is the limit of μ_n as $n \to \infty$. The above integral can be computed explicitly for equally spaced points on $[-1,1]$, where the density function is $\mu = \frac{1}{2}$ and the potential function for $z = x + iy$ is

$$\delta(z) = \frac{1}{2}\int_{-1}^{1} \log|z-t|\, dt = \frac{1}{2}\int_{-1}^{1} \log\sqrt{(x-t)^2 + y^2}\, dt$$
$$= \tfrac{1}{4}(1-x)\log\left((x-1)^2 + y^2\right) + \tfrac{1}{4}(1+x)\log\left((x+1)^2 + y^2\right)$$
$$- \tfrac{1}{2}\, y \arctan\tfrac{x-1}{y} + \tfrac{1}{2}\, y \arctan\tfrac{x+1}{y}.$$

Hence

$$e^{\delta(z)} = e^{-1}((x-1)^2 + y^2)^{(1-x)/4}((x+1)^2 + y^2)^{(1+x)/4}$$
$$\times \exp\left[\frac{1}{2} y\left(\arctan\frac{x+1}{y} - \arctan\frac{x-1}{y}\right)\right].$$

For real part x, we have

$$e^{\delta(x)} = e^{-1}(x-1)^{(1-x)/2}(x+1)^{(1+x)/2},$$

where the real part of $\delta(z)$

$$\delta(z) = -1 + \Re\left(\frac{z+1}{2}\log(z+1) + \frac{z-1}{2}\log(z-1)\right).$$

We have $\delta(0)$ = -1 and $\delta(\pm 1) = -1 + \log 2$. Then,

$$|\phi_n(z)| = \exp\big(n\,\delta(\pm 1)\big) = \left(\frac{2}{e}\right)^n \quad \text{and} \quad \exp\big(n\,\delta(0)\big) = \left(\frac{1}{e}\right)^n.$$

For the imaginary part y, we have

$$\mathbf{e}^{\delta(iy)} = e^{-1}\sqrt{1+y^2}\, \exp\left(y \arctan\frac{1}{y}\right) = \frac{1}{2}\log(1+y^2) + y\left(\frac{\pi}{2} - \arctan y\right) - 1.$$

If

$$\max_{x\in[-1,1]} |\delta(x)| = \frac{2}{e} = 0.7357588824 < |\delta(z)|,$$

then we have convergence on the whole interval $[-1,1]$.

The integral can also be computed explicitly for the Chebyshev points, where the density function is $\mu(x) = \frac{1}{\pi\sqrt{1-x^2}}$ and the potential function is

$$\delta(z) = \log\left|\frac{z - \sqrt{z^2-1}}{2}\right|$$

We have $\delta(0) = \log\frac{1}{2}$ and $\delta(\pm1) = \log\frac{1}{2}$. So

$$|\phi_n(z)| = \exp(n\delta_n) \approx 2^{-n} \quad \text{on } [-1,1].$$

Therefore, from the above we can see the difference between the values of $|\phi_n(z)|$ at equally spaced points at the ends and in the middle of the interval. On the other hand the value of $|\phi_n(z)|$ at Chebyshev points is the same. Therefore, the error increases exponentially as n increases for equally spaced points whereas the error decrease exponentially for Chebyshev points. Note that, the convergence depends on $\left|\frac{\delta(x)}{\delta(iy)}\right|^{n+1}$ as $n \to \infty$.

From the above and Lemma.1, the interpolating polynomial converges exponentially if the function f has poles in the region with $\delta(x) - \delta(z) < 0$ and diverges if f has poles in the region with $\delta(x) - \delta(z) > 0$ in some region outside the interval.

Proposition.1: *Let f be an analytic function on $[-1,1]$ and p_n be a sequence of polynomials interpolating f at equally spaced points $\{x_j\}_{j=0}^{n}$. Let $\delta(z)$ be defined as above and let C_ρ be the boundary of $\delta(z)$ in the complex plane, which is given by*

$$C_\rho = \{x + iy: \delta(x) = \delta(iy) = \rho\} \qquad (12.4)$$

Then

- If f is analytic for all z such that $\delta(z) < \delta(\rho)$, then the interpolant converges to f on $[-1,1]$.
- If f is not analytic for all z and has a pole z^*, and z is such that $\delta(z) > \delta(z^*)$, then the interpolant does not converge to f on $[-1,1]$

If $f(x)$ has singularities only outside the region of convergence, then the interpolation will converge everywhere on $[-1,1]$.

The level curve C_ρ as defined in (12.4) which go around the interval $[-1,1]$ as the number of interpolation points n tend to ∞ is very important.

We summarize our observations in the following proposition :

Proposition.2: Let we

- *Consider the analytic function $f(x) = \frac{1}{x-r}$ on $[-1,1]$ where r is real and does not lie on the interpolation interval. Then the level curve is given by*
  $\delta(z) = lim_{x\to1}\,\delta(x) = 2\ log\,2 - 2$, *where* $\delta(x) - \delta(r) < 0$. *Hence, the interpolating polynomial converge to the function (see Figures 9&10).*

- *Consider the analytic function* $f(x) = \frac{1}{s^2 + x^2}$, *where* $s > 0$ *is real on* $y - axis$. *The poles of this function are* $\pm\, is$ *and so* $\delta(x) - \delta(is) < 0$. *Therefore,* p_n*converges if* $s > 0.5$ *and diverges if* $0 < s < 0.5$.

Figures (9) and (10) show the polynomials interpolating the function $\mathrm{f(x)} = \frac{1}{\mathrm{x-r}}$ for different r

This function can be written as$\mathrm{f(x)} = \frac{1}{\mathrm{r}}\left(\frac{1}{1-\frac{\mathrm{x}}{\mathrm{r}}}\right)$. Since $|\mathrm{x}| < 1$, the Taylor series for this function

$$\mathrm{f(x)} = \frac{1}{\mathrm{r}}\sum_{\mathrm{k}=0}^{\infty}\left(\frac{\mathrm{x}}{\mathrm{r}}\right)^{\mathrm{k}},$$

which converges if $\left|\frac{\mathrm{x}}{\mathrm{r}}\right| \le \left|\frac{1}{\mathrm{r}}\right|$.

Notice that this is an improvement on the bound $\mathrm{s} > |\mathrm{b}-\mathrm{a}| = 2$ from Theorem.3.

Remark.1 [15]: Let f be an analytic function in the ellipse E_ρ interpolated by polynomials at Chebyshev points. Then

- If f has real singularities at $\mathrm{x} = \pm\mathrm{r}$, where $\mathrm{r} > 1$, then the corresponding radius of the ellipse is $\rho = \mathrm{r} + \sqrt{\mathrm{r}^2 - 1}$
- If f has imaginary singularities at $= \pm\mathrm{is}$, where $\mathrm{s} > 0$, then the ellipse has radius $\rho = \mathrm{s} + \sqrt{\mathrm{s}^2 + 1}$.

The remark shows that, if the region is large, then the convergence is faster.

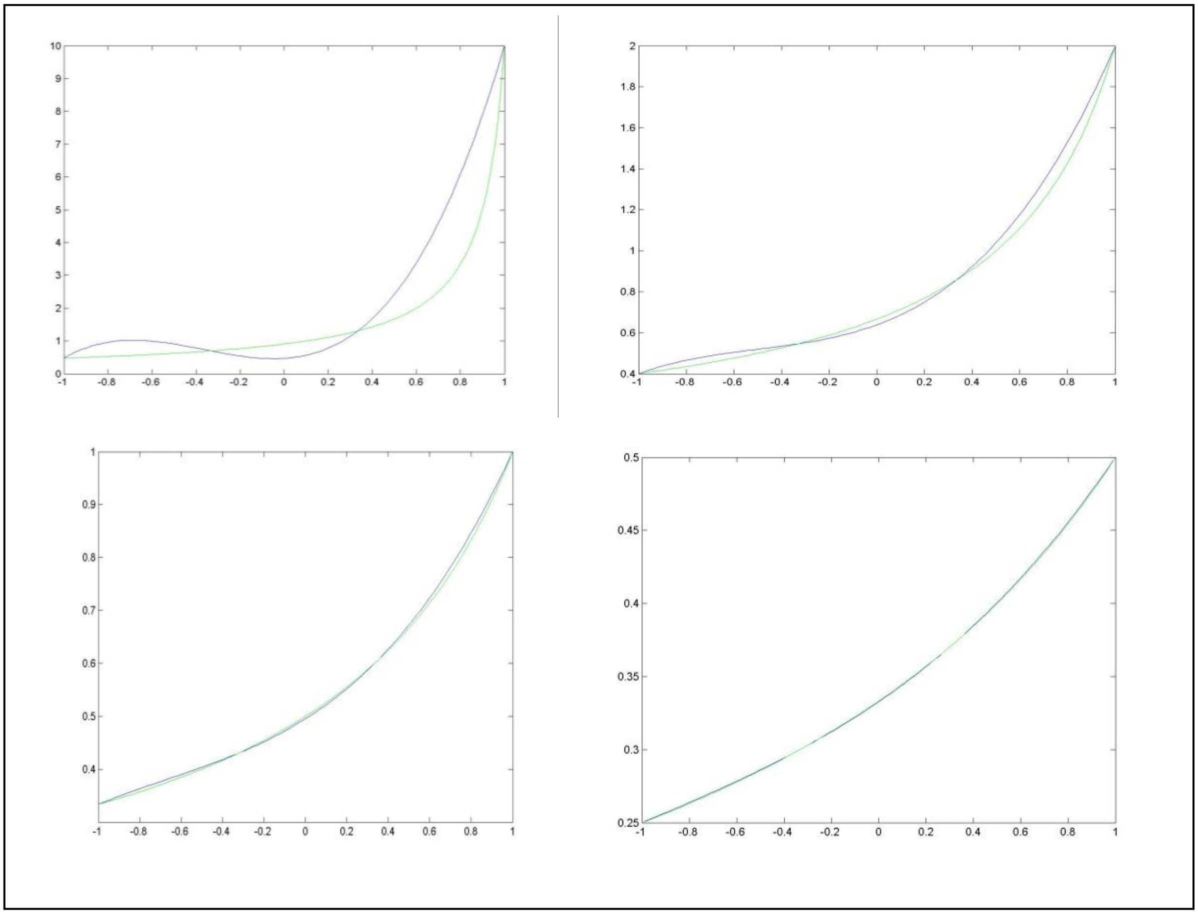

Figure 9. Polynomial interpolation on $[-1,1]$ for the function $f(x) = \frac{1}{x-r}$ at four equally spaced points for different values of r. Clockwise from the top-left: $r = 1.1$, $r = 1.5$, $r = 3$ and $r = 2$.

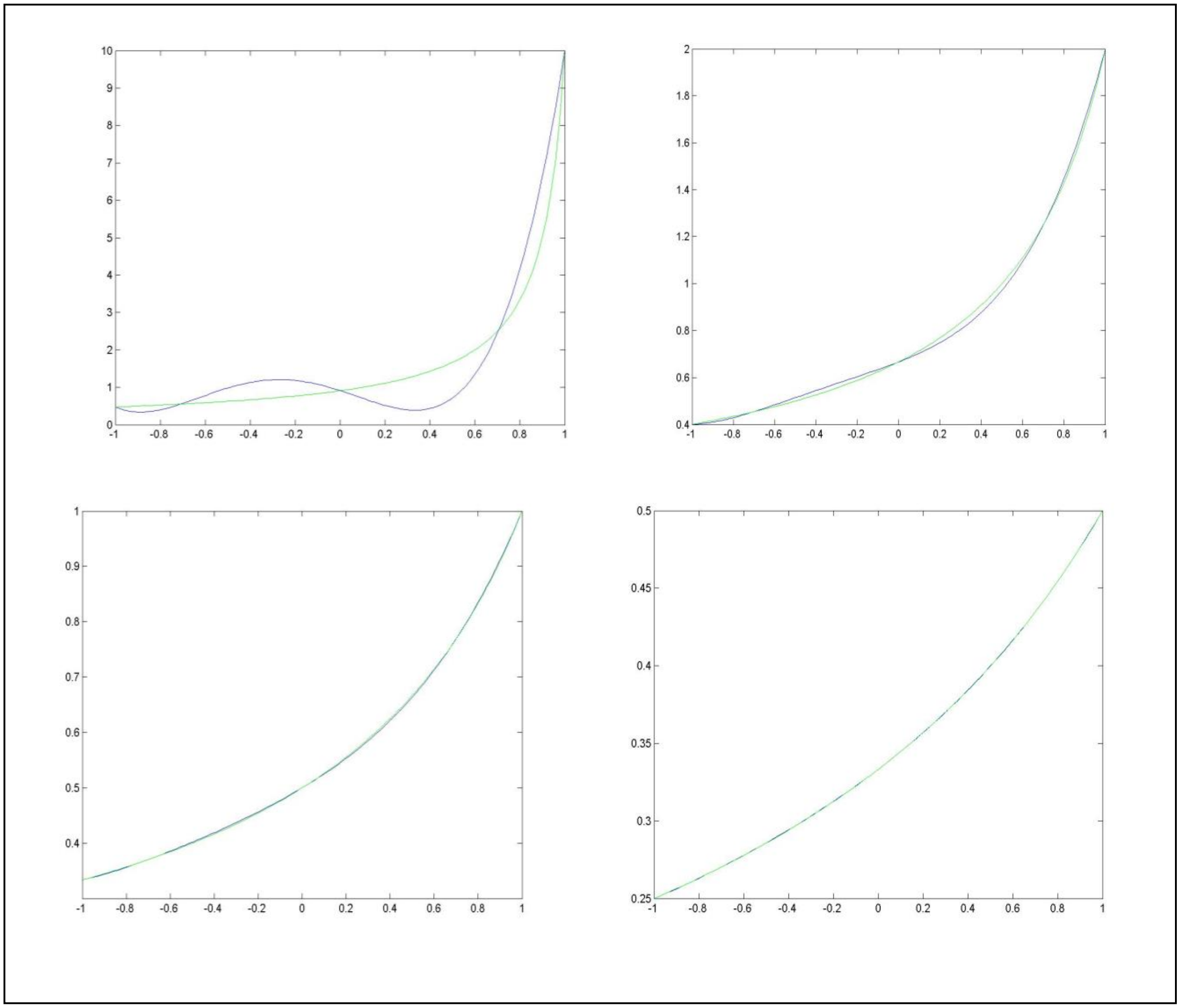

Figure10. Polynomial interpolation on $[-1,1]$ for the function $f(x) = \frac{1}{x-r}$ at four Chebyshev points for different values of r. Clockwise from the top-left: $r = 1.1$, $r = 1.5$, $r = 3$ and $r = 2$.

Example.3:

- *The analytic function* $f(x) = \frac{1}{2-x}$ *has pole at* 2 *and* $\rho = 2 + \sqrt{2^2 - 1} \approx 3.732\ldots\ldots$ *We would expect fast convergence.*
- *The analytic function* $f(x) = \frac{1}{1+ 16x^2}$ *has poles at* $\pm i/4$ *and* $\rho = \frac{1}{4} + \frac{\sqrt{17}}{4} \approx 1.28\ldots$ *We would expect slow convergence.*

Example.4: Reconsider the previous example

$$f(x) = \frac{1}{1 + x^2}, \qquad x \in [-5, 5]$$

with poles at $\pm i$. The logarithm potential for this is given by

$$\delta(z) = \int_{-5}^{5} \ln|z - t|\ dt.$$

As $\delta(z)$ is symmetric with respect to the real axis and the poles of the function are $x = \pm i$, the region of convergence is given by

$$C_\rho = \{z \in \mathbb{C},\ \ \delta(z) < \delta(i)\},$$

where the boundary of the region (the curve $\delta(z) = \delta(i)$) cuts the real axis at $x^* = 3.6338430238$. Thus p_n converge to f for $|x| < 3.633$ and diverge in the interval$(-5, x^*) \cup (x^*, 5)$, except at the points ± 5, since these are the interpolation points. This function and all its derivatives are continuous and bounded for all real x, but it has poles in the complex plane. However, these poles are too close to the interval$[-5{,}5]$. We can rewrite the function as

$$f(z) = \frac{i/2}{z+i} - \frac{i/2}{z-i}$$

which has residues of $i/2$ at $z = -i$ and $-i/2$ at $z = i$. Therefore, by the residue theorem

$$f(x) - p_n(x) = \frac{-i\emptyset_n(x)}{2\emptyset_n(-i)(-i-x)} + \frac{i\emptyset_n(x)}{2\emptyset_n(i)(i-x)}$$

For even n and equally spaced points in $[-5{,}5]$, the function $\emptyset_n(x)$ is even, which means that $\emptyset_n(i) = \emptyset_n(-i)$ and so we have

$$\frac{\emptyset_n(x)}{\emptyset_n(i)}\left[\frac{i/2}{i-x} + \frac{i/2}{i+x}\right] = \frac{-\emptyset_n(x)}{\emptyset_n(i)(1+x^2)}$$

Therefore the convergence depends on the behavior of $\frac{\emptyset_n(x)}{\emptyset_n(i)}$. Now

$$|\emptyset_n(x)|^{1/n} = \frac{1}{n}\log|\emptyset_n(x)| = \frac{1}{n}\sum_{k=0}^{n}\log|x - x_k|,$$

where

$$\lim_{n\to\infty}\frac{1}{n}\log|\emptyset_n(x)| = \frac{1}{10}\int_{-5}^{5}\log|x-t|dt = q(x).$$

The function $q(x)$ is real for complex x, and the equation ($q(x) = q(i)$) has real roots $x = \pm x^*$, where x^*=3.63.... If $q(i) = \delta(i)$, then for $x > x^*$ we have $q(x) = \delta(x) > \delta(i)$. Now

$$\log|\emptyset_n(x)|^{1/n} \to \delta(x) \qquad \text{and} \qquad \log|\emptyset_n(i)|^{1/n} \to \delta(i)$$

Then, for large n

$$\left|\log|\emptyset_n(x)|^{1/n} - \log|\emptyset_n(i)|^{1/n}\right| > |\delta(x) - \delta(i)|.$$

In other words,

$$\left|\frac{\emptyset_n(x)}{\emptyset_n(i)}\right| > e^{(\delta(x) - \delta(i))n}.$$

When $|x| > x^*$, since $\delta(x) > \delta(i)$, the error converges to infinity and therefore the function diverges for $|x| > x^*$. The curve with ρ= 2.46879067 passes through the poles $x = \pm i$ of the function and cuts the real line ± 3.6333843. Hence, the sequence p_n converges inside this curve. For Chebyshev points we have

$$d\mu = \frac{dt}{\pi\sqrt{25 - t^2}} \quad \text{and} \quad \delta(z) = \frac{2}{z + (z^2 - 25)^{1/2}}.$$

Therefore, we have, $e^{-\delta(z)} = 2.5$, and its potential curve for values larger than 2.5 are ellipses with foci ± 5. The Runge function can be expanded as a power series centred at the origin

$$\frac{1}{1 + x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}, \quad |x| < 1.$$

The series converges inside the unit disk and diverges outside of it.

For a generalization of the Runge function, we have the following:

Proposition.3: *Consider a generalization* $f(x) = \frac{1}{s^2 + x^2}$ *of Runge's example on the interval* $[-a, a]$, *where* $s > 0$ *is on* $y - axis$. *Then*

$$|f(x) - p_n(x)| \le \frac{|r|}{s^2 + x^2} \left| \frac{\emptyset_n(x)}{\emptyset_n(si)} \right|.$$

where $x = r$ for even n and $r = si$ for odd n. The convergence of p_n depends on $\left|\frac{\emptyset_n(x)}{\emptyset_n(si)}\right|$. If

$$\max_{x\in[-a,a]} \delta(x) < \delta(si)$$

where

$$\max_{x\in[-a,a]} \delta(x) = \delta(a) = \frac{2a}{e}$$

and $\delta(si) \le \delta(x)$. Therefore the convergence occurs when $s > \zeta^* a$ where ζ^*=0.5255. (See also [7]).

6. Conclusion

We can conclude that, if the interpolated function f is real analytic and its poles in the complex plane are located far from the interpolation interval, then interpolation converges uniformly to f. If the above condition is not satisfied, the polynomial interpolation may diverge for certain distributions of points. In particular, rational functions with singularities near the interpolation interval may lead to the Runge phenomenon

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