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حول تذبذبات المعادلات التفاضلية المخمدة غير الخطية من الدرجة الثانية

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الملخص

تقدم هذه المقالة معايير جديدة لتحليل الأنماط التذبذبية التي تظهرها حلول المعادلات التفاضلية المخمدة. وقد استخلصنا النتائج الرئيسية من خلال استخدام تقنيات حاسمة، مثل طريقة ريكاتي والمتوسط التكاملي. تقدم النظريات المقدمة في الدراسة نهجا أكثر شمولا وتوسعا مقارنة بالأبحاث السابقة. وقد تم تضمين أمثلة توضيحية لتسليط الضوء على الفروق بين نتائجنا وتلك الموثقة في الأدبيات الموجودة.

الكلمات المفتاحية

السلوك التذبذبي. المعادلات التفاضلية المخمدة. الدرجة الثانية.

On Oscillation of Second Order Damped Nonlinear Differential Equations

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Abstract

This article introduces novel criteria for analyzing the oscillatory patterns exhibited by the solutions of damped differential equations. The main results we are derived through the utilization of crucial techniques, such as the Riccati method and integral averaging. The theorems presented in this study offer a more comprehensive and extended approach than compared to prior research. Illustrative examples are included to highlight the distinctions between our findings and those documented in existing literature.

Keywords

oscillatory behaviour. damped differential equations. second order.

1. Introduction and Preliminaries

Consider the following differential equation of the form:

$$(r(t)f(x'(t)))' + h(t)f(x'(t)) + q(t)g(x(t)) = H(t,x'(t),x(t)). \qquad \text{(E)}$$

where $r, q \text{ and } h \in C\,[t_0,\infty), t_0 \geq 0,\; r(t) > 0$ and $g(x) \in C^1(\mathbb{R})$ except at 0.

The study of oscillations in differential equations is an essential aspect of the analysis o f differential equations, with many applications in engineering and the natural sciences. The vib

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ration characteristics of various components have attracted considerable interest, leading to ext ensive research on oscillatory models in various types of differential equations. Many researche s have examined the oscillatory behavior of solutions of differential equations (E).For example, Elabbasy et al. (2005), established new oscillation conditions for Kamanev and Philos type for all solutions of Eq. (E) with $f(x') = x'$and $H(t, x'(t), x(t)) = 0$ and they do not sign the conditions on $h(t), q(t)$ as follows:

$$(r(t)x'(t))' + h(t)x'(t) + q(t)g(x(t)) = 0. \qquad (E_1)$$

Zhang and Song (2011), considered equation (E) and replaced the explicit function $q(t)g(x(t))$ by the implicit function $Q(t, x)$, and obtained some enough oscillation criteria for solutions of (E).

Wang and Song (2013), discussed the oscillation conditions of (E) with $H\big(t, x'(t), x(t)\big) = 0$and $\frac{g(x)}{x} \geq k > 0$ for $x \neq 0$ as following:

$$(r(t)\psi(x(t))x'(t))' + h(t)x'(t) + Q(t, x) = H(t, x(t), x'(t)) \qquad (E_2).$$

Moreover, Salhin et al. (2014), studied a class of equations more general than (E) where she employed a class of functions to obtain new oscillation conditions for the monotonicity function for previous results in the literature as follows:

$$(r(t)\psi(x(t))f(x'(t)))' + h(t)f(x'(t)) + q(t)g(x(t)) = H(t, x(t), x'(t)) \qquad (E_3).$$

Zhang et al. (2016), obtained two new oscillation conditions using a generalized Riccati transform and an integral averaging technique of the Philos type.

More recently, Salhin (2023), derived new sufficient conditions for the oscillation of the solutions to Eq. (E) with positive functions $\psi\big(x(t)\big)$ and $f\left(x'(t)\right) = x'(t)$.

In fact, many new ideas for determining sufficient conditions for oscillations can be found in the papers of Lu et al., (2007), Rogovchenko and Tuncay, (2007), Rogovchenko and Tuncay, (2008), Rogovchenko and Tuncay, (2009), Tunc and Avci, (2012) and Wang and Song, (2011)) and the recent monograph of Mazen et al., (2024). Those references and others will be considered in order to come out with new results regarding our subject.

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The purpose of this study, is to contribute further in this direction and to establish sufficient conditions for Eq. (E).

A solution to (E) is called oscillatory if it is neither eventually negative nor eventually positive. The differential equation is oscillatory if all its solutions are oscillation.

2. Main Results

With respect to (E), we need to suppose that there are the positive constants e_1, e_2, l and k satisfy:

(1) $l > 0$and $f^2(y) \leq l y f(y)$forall$y \in \mathbb{R}$,

(2) $q(t) \geq 0$,

(3) $x g(x) > 0$and $0 < k \leq g^{'}(x)$ for all$x \neq 0$,

(4) $p: [t_0, \infty) \to \mathbb{R}$is continuous function such that $\frac{H(t,x,y)}{g(x)} \leq p(t) \forall t \in t_0, \infty)$; $x, y \in \mathbb{R}$ and $x \neq 0$,

(5) $0 < e_1 \leq \frac{f(y)}{y} \leq e_2$forall $y \neq 0$.

The following lemma will be need to prove the main results. First, Defined the continuous functions as $h_3, H: D = \{(t,s): t_0 \leq s \leq t\}$. A function $H \in (D, \mathbb{R})$is said to belong to class ξif

i. $H(t,t) = 0$ for $t \geq t_0$and $H(t,s) > 0$ when $t \neq s$,

ii. $H(t,s)$ has partial derivatives on D such that

$$\frac{\partial H(t,s)}{\partial s} = -h_3(t,s)\sqrt{H(t,s)},$$

$\frac{\partial H(t,s)}{\partial t} = h_4(t,s)\sqrt{H(t,s)}$,for some$h_3, h_4 \in L^1_{loc}(D, \mathbb{R})$

Lemma 1. Let$A_1, A_2, A_3 \in C([t_0, \infty), \mathbb{R})$with$A_3 > 0, Z \in C^1([t_0, \infty), \mathbb{R})$**.** If there exist $(a,b) \subset [t_0, \infty)$ *and* $c \in (a,b)$ *such that*

$$Z^{'} \leq -A_1(s) + A_2(s)\, Z - A_3(s)\, z^2 \; s \in (a,b),$$

then

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$$\frac{1}{H(c,a)}\int_a^c\left[H(s,a)A_1(s)-\frac{1}{4A_3(s)}\theta_1^2(s,a)\right]ds$$
$$+\frac{1}{H(b,c)}\int_c^b\left[H(b,s)A_1(s)-\frac{1}{4A_3(s)}\theta_2^2(b,s)\right]ds\leq 0$$

for all $H\in\xi$, where

$$\theta_1(s,a)=\left[h_3(s,a)+A_2(s)\sqrt{H(s,a)}\right]$$
$$\theta_2(b,s)=\left[h_4(b,s)-A_2(s)\sqrt{H(b,s)}\right].$$

The proof of this lemma Can be found in Lu and Meng [2].

In the next theorems we put:

$$\beta(t)=e_2\phi'(t)r(t)-e_1\phi(t)h(t),$$
$$\delta(t)=\frac{l}{\phi(t)r(t)},$$
$$v[t,T]=\delta(t)\left(\int_T^t\delta(s)ds\right)^{-1}$$

Theorem 1. Suppose that $(1)-(5)$ true. Assume that

$$\int^{\infty}\frac{du}{g(u)}<\infty,\int^{-\infty}\frac{du}{g(u)}<\infty, \qquad (6)$$

$$\int^{\infty}\frac{\sqrt{g'(u)}}{g(u)}du<\infty,\int^{-\infty}\frac{\sqrt{g'(u)}}{g(u)}du<\infty \qquad (7)$$

$$min\left\{\inf_{u>0}\sqrt{g'(u)}\int_u^{\infty}\frac{\sqrt{g'(u)}}{g(u)}du,\inf_{u<0}\sqrt{g'(u)}\int_u^{-\infty}\frac{\sqrt{g'(u)}}{g(u)}du\right\}>0 \qquad (8)$$

$$\beta(t)\geq 0,\beta'(t)\leq 0,t\geq t_0, \qquad (9)$$

$$\int^{\infty}\delta(s)ds=\infty, \qquad (10)$$

There exists a continuously differentiable function $\phi\in C^1[t_0,\infty)\to(0,\infty)$, such that $\phi'\geq 0$ and $\acute{\phi}'\leq 0$, we have

$$\lim_{t\to\infty}\inf\int_{t_0}^t\phi(s)(q(s)-p(s))ds>-\infty. \qquad (11)$$

There exists $(a,b)\subset[T,\infty)$, $c\in(a,b)$, $H\in\xi$, and for a constant $F>0$, such that

$$\frac{1}{H(c,a)}\int_a^c[H(s,a)\phi(s)(q(s)-p(s))-\frac{1}{4Jv[s,t_0]}\theta_1^2(s,a)]ds$$

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$+\frac{1}{H(b,c)}\int_c^b[H(b,s)\phi(s)(e_3q(s)-p(s))-\frac{1}{4Jv[s,t_0]}\theta_2^2(b,s)]ds>0,$ (12)

where

$$\theta_1(s,a)=\left[h_3(s,a)+\frac{1}{e_1}\delta(s)\beta(s)\sqrt{H(s,a)}\right],$$

$$\theta_2(b,s)=\left[h_4(b,s)-\frac{1}{e_1}\delta(s)\beta(s)\sqrt{H(b,s)}\right].$$

Then all solutions of (E) are oscillates.

Proof. Assume that $x(t)$ a non-oscillatory solution of (E) such that $x(t)>0$ on $[T,\infty)$ for some $0\le t_0\le T$. Letting

$$w(t)=\frac{\phi(t)r(t)f(x'(t))}{g(x(t))}. \qquad (13)$$

Differentiating (13), using Eq. (E) and (1) – (5) we obtain

$$w'(t)\le\phi(t)\big(q(t)-p(t)\big)-\frac{e_1\phi(t)h(t)x'(t)}{g(x(t))}+\frac{e_2\phi'(t)r(t)x'(t)}{g(x(t))}$$

$$-\frac{l}{\phi(t)r(t)}w^2(t)g'\big(x(t)\big).$$

$$=-\phi(t)q(t)-p(t))+\big(e_2\phi'(t)r(t)-e_1\phi(t)h(t)\big)\frac{x'(t)}{g(x(t))}-\frac{l}{\phi(t)r(t)}w^2(t)g'(x(t))$$

$$=-\phi(t)(q(t)-p(t))+\beta(t)\frac{x'(t)}{g(x(t))}-\delta(t)w^2(t)g'(x(t)). \qquad (14)$$

Integrating (14) from t_0 to $t$we get that

$$w(t)\le w(t_0)-\int_{t_0}^t\phi(s)\,(q(s)-p(s))\,ds+\int_{t_0}^t\beta(s)\frac{x'(s)}{g(x(s))}ds$$

$$-\int_{t_0}^t\delta(s)\,w^2(s)\,g'(x(s))ds. \qquad (15)$$

Since $\beta'(s)\le 0$, then there exist $b_1\in[t_0,\infty)$foreevery$t\ge t_0$such that

$$\int_{t_0}^t\beta(s)\frac{x'(s)}{g(x(s))}ds=\beta(t_0)\int_{t_0}^{b_1}\frac{x'(s)}{g(x(s))}ds=\beta(t_0)\int_{x(t_0)}^{x(b_1)}\frac{du}{g(u)}$$

$$\le\beta(t_0)\int_{x(t_0)}^{\infty}\frac{du}{g(u)}=e_3, \qquad (16)$$

where $e_3>0$ is a constant. Then, we have for $t\ge t_0$,

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$w(t) \le F - \int_{t_0}^{t} \phi(s)\,(q(s) - p(s))\,ds - \int_{t_0}^{t} \delta(s)\,w^2(s)\,g'(x(s))\,ds,$ (17)

where $F = w(t_0) + e_3$.

We have one of the following three cases:

Case 1. If $x'(t)$ is oscillates, choose $t_n \ge t_1$ such that $\lim_{n\to\infty} t_n = \infty$ and $x(t_n) = 0, n = 1,2,\ldots$ on $[t_0,\infty)$. From (17) we get

$$\int_{t_0}^{t_n} \delta(s)\,w^2(s)\,g'(x(s))\,ds \le F - \int_{t_0}^{t_n} \phi(s)\,(q(s) - p(s))\,ds, n = 1,2,\ldots$$

Using (11) we obtain

$$\int_{t_0}^{t_n} \delta(s)\,w^2(s)\,g'(x(s))\,ds < \infty.$$

There exists $A > 0$, such that

$\int_{t_0}^{t_n} \delta(s)\,w^2(s)\,g'(x(s))\,ds \le A, t \ge t_0.$ (18)

Using Schwarz inequality, (5) and (18) we have

$$\left|\int_{t_0}^{t} \frac{x'(s)}{g(x(s))}\sqrt{g'(x(s))}ds\right|^2 = \frac{1}{e_1}\left|\int_{t_0}^{t} \sqrt{\delta(s)}\left(\sqrt{\delta(s)}w(s)\sqrt{g'(x(s))}\right)ds\right|^2,$$

$$\le \frac{1}{e_1}\left(\int_{t_0}^{t} \delta(s)ds\right)\left(\int_{t_0}^{t_n} \delta(s)\,w^2(s)\,g'(x(s))\,ds\right),$$

$\le \frac{A}{e_1}\int_{t_0}^{t} \delta(s)ds\,, t \ge t_0.$ (19)

Applying (8),

$\sqrt{g'(x(t))}\int_{x(t)}^{\infty} \frac{\sqrt{g'(u)}}{g(u)}du \ge M, t \ge t_0,$ (20)

where $M$is a positive constant.

Let $M_1 = \int_{x(t_0)}^{\infty} \frac{\sqrt{g'(u)}}{g(u)}du > 0$,and applying (20) we have

$$g'(x(t)) \ge M^2\left[\int_{x(t)}^{\infty} \frac{\sqrt{g'(u)}}{g(u)}du\right]^{-2} = M^2\left[M_1 - \int_{x(t_0)}^{x(t)} \frac{\sqrt{g'(u)}}{g(u)}du\right]^{-2},$$

$$= M^2\left[M_1 - \int_{t_0}^{t} \frac{x'(s)}{g(x(s)))}\sqrt{g'(x(s))}ds\right]^{-2},$$

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$$\geq M^2\left[M_1+\left|\int_{t_0}^{t}\frac{x'(s)}{g(x(s)))}\sqrt{g'(x(s))ds}\right|\right]^{-2}.$$

Using (19) in the above inequality leads to

$$g'(x(t))\geq M^2\left[M_1+\left(\frac{A}{e_1}\int_{t_0}^{t}\delta(s)ds\right)^{\frac{1}{2}}\right]^{-2}.$$

Then, there exists a constant $J>0$and$T>t_0$, such that

$$g'(x(t))\geq J\left(\frac{A}{e_1}\int_{t_0}^{t}\delta(s)ds\right)^{-1}, t\geq T. \qquad (21)$$

Substituting of (21) in (14) we get

$$w'(t)\leq -\phi(t)(q(t)-p(t))+\frac{1}{e_1}\delta(t)\beta(t)w(t)-Jv[t,t_0]w^2(t), t\geq T. \qquad (22)$$

From (22) and by Lemma 1 we conclude that for any$c\in(a,b)$ and$H\in\xi$

$$\frac{1}{H(c,a)}\int_a^c\left[H(s,a)\phi(t)(q(t)-p(t))-\frac{1}{4Jv[t,t_0]}\theta_1^2(s,a)\right]ds$$
$$+\frac{1}{H(b,c)}\int_c^b\left[H(b,s)\phi(t)(q(t)-p(t))-\frac{1}{4Jv[t,t_0]}\theta_2^2(b,s)\right]ds\leq 0,$$

which contradicts the condition (12).

Case 2. Assume that $x'(t)>0$for$t_0\leq t_1\leq t$, then $w(t)>0$for$t\geq t_1$, by (17) we have

$$\int_{t_1}^{t}\delta(s)w^2(s)g'(x(s))\,ds\leq F-\int_{t_1}^{t}\phi(s)(q(s)-p(s))\,ds, t\geq t_1.$$

From (11) we see that

$$\int_{t_1}^{\infty}\delta(s)w^2(s)g'(x(s))\,ds<\infty. \qquad (23)$$

The proof of the following case will be was similar to Case1.

Case 3. Let $x'(t)<0$ for $t_0\leq t_1\leq t$, If (23) holds, then we have same the discussion in Case 2. If (23) is failed, by (17) and (11) we can get:

$$F_1+\int_{t_1}^{t}\delta(s)w^2(s)g'(x(s))\,ds\leq -w(t), t\geq t_1, \qquad (24)$$

where F_1 is a constant. By taking $t_2\geq t_1$,

$$F_2=F_1+\int_{t_1}^{t_2}\delta(s)w^2(s)g'(x(s))\,ds>1. \qquad (25)$$

From (24) and (25)

$$w(t)<0,$$

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and from (23), we find

$$\frac{\delta(t)w^2(t)g^{'}(x(t))}{F_1+\int_{t_1}^{t}\delta(s)w^2(s)g^{'}(x(s))\,ds}\geq\frac{-x^{'}(t)g(x^{'}(t))}{g(x(t))},t\geq t_2.$$

Integrating the above inequality, we have

$$ln\left[F_1+\int_{t_1}^{t}\delta(s)w^2(s)g^{'}(x(s))\,ds\right]\geq ln\frac{g(x(t_2))}{g(x(t))},t\geq t_2.$$

Therefore

$F_1+\int_{t_1}^{t}\delta(s)w^2(s)g^{'}(x(s))\,ds\geq\frac{g(x(t_2))}{g(x(t))},t\geq t_2.$ (26)

Applying (24) and (26), we obtain

$$x^{'}(t)\leq-\frac{1}{e_1}\delta(t)g(x(t_2))<0,t\geq t_2.$$

Hence

$$x(t)\leq x(t_2)-\frac{1}{e_1}g(x(t_2))\int_{t_2}^{t}\delta(s)ds\to-\infty,t\to\infty.$$

Which is a contradiction. Hence the proof is completed.

□

Theorem 2. In addition to the conditions (1) − (5) and (6) – (8), assume that the function $\phi:[t_0,\infty)\to(0,\infty)$, such that (9) – (12) satisfied, and $H\in\xi$, such that

$\limsup_{t\to\infty}\int_{a_1}^{t}[H(s,a_1)\phi(s)(q(s)-p(s))-\frac{1}{4Jv[s,t_0]}\theta_1^2(s,a_1)]ds>0.$ (27)

Then Eq. (E) is oscillatory if

$\limsup_{t\to\infty}\int_{a_1}^{t}[H(t,s)\phi(s)(q(s)-p(s))-\frac{1}{4Jv[s,t_0]}\theta_2^2(b,s)]ds>0$ (28)

true for all $a_1\in[t_1,\infty)$, where θ_1,θ_2 and J as in Theorem 1.

Proof. If $x(t)\neq0\ \forall\ t\in[t_2,\infty)$ for some $t_2\geq t_1$. Put $a_1=a\geq t_2$ in (27). We can get $c>a$ such that

$\int_{a}^{c}[H(s,a)\phi(s)(q(s)-p(s))-\frac{1}{4Jv[s,t_0]}\theta_1^2(s,a)]ds>0.$ (29)

Similarly, with (28) by setting $a_1=c\geq t_2$. This leads to exists $b>c$ such that

$\int_{c}^{b}[H(b,s)\phi(s)(q(s)-p(s))-\frac{1}{4Jv[s,t_0]}\theta_1^2(b,s)]ds>0.$ (30)

We can note that, (12) is true. Then the solutions of Eq. (E) are oscillates.

□

Example 1. Consider the damped differential equation:

$$\left[\left(\frac{1}{1+t^2}\right)\left(3x'(t)\frac{(x'(t))^7}{1+(x'(t))^6}\right)\right]' - \frac{1}{t}\left(3x'(t)\frac{(x'(t))^7}{1+(x'(t))^6}\right)$$
$$+\frac{3+t^2}{4}\left[\frac{2}{t-(6n-4)\pi}+\frac{1+t^2}{t}\right]x(t)(1+(x(t))^2)=\frac{x^3(t)\cos t\sin x'(t)}{t^2},$$

$(6n-4)\pi \le t \le (6n-\frac{7}{2})\pi$ for $n=1,2,\ldots$

From Example 1. we can see that

$r(t)=\frac{1}{1+t^2}, h(t)=-\frac{1}{t}$,when$t \ge t_0=\frac{\pi}{2}$.

Let $\phi(t)=1$, we can see that

$$\beta(t)=e_2\phi'(t)r(t)-e_1\phi(t)h(t)=\frac{11}{t},$$

$$\delta(t)=\frac{l}{\phi(t)r(t)}=1+t^2,$$

$$v[t,\frac{\pi}{2}]=\delta(t)\left(\int_T^t \delta(s)ds\right)^{-1}=\frac{1+t^2}{t+\frac{t^3}{3}+\frac{\pi}{2}+\frac{\pi^3}{24}}.$$

Also, we notice that the conditions (9) and (10) of Theorem 1 are satisfied by using $\beta(t)$ and $\delta(t)$ respectively.

Remark 1. Theorems 1. and Theorem 2. include theorems 1 and 2 of Lu and Mang, (2007) and generalize, improve and unify the results of Zhang et al., (2024) with $f(x')=x'$ and $H(t,x'(t),x(t))=0$.

3. Conclusions

In conclusion, we have established and demonstrated a new set of oscillation conditions that improve and extend the existing oscillation criteria, treating cases not previously covered by known results. In addition, we have provided illustrative examples to support our work. We have also included notes to highlight the importance of our main findings.

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