Solution of Abel's Integral Equation By Using Sawi Integral Transform

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Abstract: In this work, we introduce a new application for solving a singular integral equation; specifically**,** Abel's integral equation of the first type using Sawi integral transform. We provide a number of examples of demonstrate the effectiveness of the suggested approach. Results indicate that, the Sawi integral transformation is a very helpful and efficient integration tool for solving and simplifying this type of integral equation.

Keywords: Abel's Integral Equation, Convolution, Sawi Transformation, Inverse Sawi Transformation.

حل معادلة آبل التكاملية باستخدام التحويل التكاملي ساوي

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المستخلص: في هذا العمل، نُقدّم تطبيقًا جديدًا لحل معادلة تكاملية شاذة، وتحديدًا معادلة آبل التكاملية من النوع الأول، باستخدام تحويل ساوي التكاملي. نُقدّم عددًا من الأمثلة لإثبات فعالية النهج المقترح. تُظهر النتائج أن تحويل ساوي التكاملي أداة تكامل فعّالة ومفيدة للغاية لحل وتبسيط هذا النوع من المعادلات التكاملية.

الكلمات المفتاحية: معادلة ابل التكاملية، الالتفاف، تحويل ساوي، تحويل ساوي العكسي.

1. Introduction:

Integral equations are among the most effective mathematical tools in the fields of pure and applied mathematics, due to their ability to formulate and model many physical and engineering phenomena. One type of integral equation is Abel's equation, which is defined as follows (Wazwaz, 2011,P36; Rahman, 2007,P98)

$$\mathrm{j}(x)=\int_0^x \frac{1}{\sqrt{x-t}}\varphi(t)\,dt \quad (1)$$

Where $\kappa(x,t)=\frac{1}{\sqrt{x-t}}$ goes to ∞ at $x=t$, there are two types of functions: known function $\mathrm{j}(x)$,and unknown function $\varphi(t)$. This equation is considered an important mathematical model widely utilized in numerous scientific domains, including X-ray radiography, radar ranging, and studying space with radio waves. (Singh, 2009, P242). This equation is characterized by the presence of a singular kernel, which makes its solution mathematically difficult and requires special techniques.

Various methods have been established for analyzing and accurately solve it. Among these methods are integral transformations; many researchers have used these integral

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transformations, such as Laplace transform (Dass, 2007, P885; Aggarwal, et al. 2018, P141; Greenberg, 1998, P247; Jeffrey, 2002, P379), Fourier transform (Lokenath & Bhatta, 2007, P639), Sumudo transform (Watugula, 1993, P35; Belgacem & Karaballi, 2006, P1; Mtawal, 2021, P5), Hankel transform (Lokenath & Bhatta, 2007, P344), Elzaki transform (Elzaki, 2011, P57; Aggarwal, et al. 2018, P281; Elzaki & Ezaki, 2011, P65, 2011,P41), Mahgoub transform (Aggarwal, et al. 2018, P173; Kiwne & Sonawane, 2018, P500 ; Mahgoub, 2019,P81), Kamal transform (Abdelilah & Hassan, 2016, P451, 2017,P7) , Aboodh transform (Aboodh, 2013, P35), Mohand transform (Mohand & Mahgoub, 2017, P113) Sawi transform (Mahgoub, 2019, P81), and others, to find solutions for integral, partial, and differential equations. Recently, the Sawi integral transformation has been developed as a new and useful instrument to solve equations of both differential, and integral types. Aggarwal & Gupta(2019, PP600-608, 2019,PP252-260, 2019,PP423-431, 2019, PP101-110) and Aggarwal & Sharma (2019, PP82-90, 2019, PP317-325) applied integral transforms to find solutions to the integral equation of Abel. The Sawi transformation was employed by Aggarwal et al.(2020, PP12-8) to evaluate Definite Integrals using Bessel's Functions. The Sawi transform was used by Singh &Aggarwal (2019, PP157-162) to address problems related to population growth and decay.

To our knowledge, there is no published study yet that addresses the application of the Sawi transform to Abel's equation. In this work, we use the Sawi transform to find the exact solution of the Abel's integral equation, and then give some applications of this tool showing its distinction.

2. Preliminaries:

This section defines the Sawi integral transform and presents some of its basic properties, which will be utilized later in processing Abel's integral equation.

2.1 Definition(Sawi transform) (Mahgoub, 2019,P81)

we consider functions to the set B defined as:

$B=\{\mathrm{j}(t):\exists N, c_1, c_2 \succ 0, \left|\mathrm{j}(t)\right| \prec Ne^{\frac{|t|}{c_m}}, t\in(-1)^m\times[0,\infty)\}$, the constant N is finite , and the constants c_1, c_2 may be finite or infinite. The Sawi transform of the function $\mathrm{j}(t)$ represented by $S[\mathrm{j}(t)]$, is defined by the following integral equation:

$$S[\mathrm{j}(t)]=\frac{1}{w^2}\int_0^{\infty}\mathrm{j}(t)e^{-(\frac{1}{w})t}dt=T(w)\,.\; t\geq 0, c_1\leq w\leq c_2 \qquad (2)$$

Where the operator S is called the Sawi transform operator.

2.2 Inverse Sawi transform (Aggarwal,et al. ,2020,P14)

Let the Sawi transform of $\mathrm{j}(x)$ is $T(w)$, then the inverse Sawi transform is

$S^{-1}[T(w)]=j(t)$.

2.3 Linearity property(Aggarwal,et al. ,2020,P13; Singh & Aggarwal, 2019,P157)

If $S[\mathrm{j}_1(t)]=T_1(w)$ and $S[\mathrm{j}_2(t)]=T_2(w)$, then

$S[c_1 j_1(t)+c_2 j_2(t)]=c_1T_1(w)+c_2T_2(w)$ where c_1, c_2 are arbitrary constants.

2.4 Sawi transformation of the derivatives of $h(x)$ (Mahgoub, 2019,P82 ; Singh & Aggarwal,2019,P158)

If $S[\mathrm{j}(t)]=T(w), then$

i. $S[j'(t)] = \frac{1}{w}T(w) - \frac{1}{w^2}j(0)$

ii. $S[j''(t)] = \frac{1}{w^2}T(w) - \frac{1}{w^2}j'(0) - \frac{1}{w^3}j(0)$

iii. $S[j^{(n)}(t)] = \frac{T(w)}{w^{(n)}} - \sum_{k=0}^{n-1}\frac{j^{(k)}(0)}{w^{n-k+1}}$

2.5 Convolution theorem for Sawi transform(Aggarwal,et al. ,2020,P13)

If If $S[j_1(t)] = T_1(w)$ and If $S[j_2(t)] = T_2(w)$ then the convolution's Sawi transformation $j_1(t)*j_2(t)$ is offered through: $S[j_1(t)*j_2(t)] = S[j_1(t)]S[j_2(t)] = w^2T_1(w)T_2(w)$, where $j_1(t)*j_2(t)$ is defined by : $j_1(t)*j_2(t) = \int_0^t j_1(t-x)\,j_2(x)\,dx = \int_0^t j_1(x)\,j_2(t-x)\,dx$

3. Solving Abel's integral equation Using the Sawi Transform

In this section, we present Sawi transform method as a useful method to solve abel's integral equation.

Using the Sawi transformation on both equation sides (1), we find

$$S[j(x)] = S[\int_0^x \frac{1}{\sqrt{x-t}}\varphi(t)dt]$$

$$\Rightarrow S[j(x)] = S[x^{-\frac{1}{2}} * \varphi(x)] \qquad (3)$$

By using the Sawi transform's convolution theorem in (3), we have

$$S[j(x)] = w^2 S[x^{-\frac{1}{2}}]S[\varphi(x)]$$

$$\Rightarrow S[j(x)] = w^2(\sqrt{\pi}\,w^{-\frac{3}{2}})S[\varphi(x)]$$

$$\Rightarrow S[\varphi(x)] = \frac{\sqrt{\pi}w^{-\frac{1}{2}}}{\pi}S[j(x)]$$

$$\Rightarrow S[\varphi(x)] = \frac{1}{w\,\pi}[w^2(\sqrt{\pi}\,w^{-\frac{3}{2}})S[j(x)]$$

$$= \frac{1}{\pi w}(w^2 S[x^{-1/2}]S[j(x)])$$

$$= \frac{1}{\pi w}S[x^{-1/2} * j(x)]$$

$$\Rightarrow S[\varphi(x)] = \frac{1}{\pi w}S\{\int_0^x \frac{1}{\sqrt{x-t}}\,j(t)dt\}$$

$$\Rightarrow S[\varphi(x)] = \frac{1}{\pi w}S\{J(x)\} \qquad (4)$$

Where $J(x)=\int_0^x \frac{1}{\sqrt{x-t}}\, j(t)dt$ (5)

Using Sawi transform of the derivative on (5), we now obtain

$$S[J'(x)]=\frac{1}{w}S[J(x)]-\frac{J(0)}{w^2},$$

$$\Rightarrow S[J(x)]=\frac{S[J(x)]}{w},$$

$$\Rightarrow S[J(x)]=wS[J'(x)], \qquad .(6)$$

From (4) and (6), we now possess

$$S[\varphi(x)]=\frac{1}{\pi}S[J'(x)], \qquad .(7)$$

The inverse Sawi transformation applied to both sides of equation (7) yields

$$\varphi(x)=\frac{1}{\pi}J'(x)=\frac{1}{\pi}\frac{d}{dx}J(x), \qquad .(8)$$

Using (5) in (8), we have

$$\varphi(x)=\frac{1}{\pi}[\frac{d}{dx}\int_0^x \frac{1}{\sqrt{x-t}}\, j(t)dt], \qquad (9)$$

This is a required solution to (1).

4. Applications:

This part includes a few mathematical examples that show how well the Sawi transform approach works to solve Abel's integral equation .

Example 4.1

Solve the Abel's integral equation.

$$x=\int_0^x \frac{1}{\sqrt{x-t}}\varphi(t)dt, \qquad (10)$$

When we take the Sawi transform of (10), we get

$$S[x]=S[\int_0^x \frac{1}{\sqrt{x-t}}\varphi(t)dt],$$

Then $1=S[x^{-\frac{1}{2}}*\varphi(x)],$ (11)

Using the Sawi transform's convolution theorem in (11), we obtain

$$1=w^2S[x^{-\frac{1}{2}}]S[\varphi(x)],$$

Then $1=w^2(\sqrt{\pi}w^{-\frac{3}{2}})S[\varphi(x)],$

$$\Rightarrow S[\varphi(x)]=\frac{w^{-\frac{1}{2}}}{\sqrt{\pi}}, \qquad (12)$$

The inverse Sawi transformation applied to both sides of equation (12) yields

$$\varphi(x)=1/\sqrt{\pi}\,S^{-1}[w^{-\frac{1}{2}}],$$

$$\Rightarrow \varphi(x)=\frac{2}{\pi}x^{\frac{1}{2}}, \qquad .(13)$$

this is the required solution to (10).

Example 4.2

Solve the Abel's equation

$$1+x+x^2=\int_0^x \frac{1}{\sqrt{x-t}}\varphi(t)dt, \qquad (14)$$

When we take the sawi transformation of (14), we get

$$S[1]+S[x]+S[x^2]=S[\int_0^x \frac{1}{\sqrt{x-t}}\varphi(t)dt],$$

Then $1/w+1+2w=S[x^{-\frac{1}{2}}*\varphi(x)], \qquad (15)$

Using the Sawi transform's convolotion theorem in (15), we have

$$1/w+1+2w=w^2S[x^{-\frac{1}{2}}]\mathrm{S}[\varphi(x)],$$

Then $1/w+1+2w=w^2(\sqrt{\pi}\,\mathrm{w}^{-\frac{3}{2}})\mathrm{S}[\varphi(x)],$

Then $S[\varphi(x)]=\frac{1}{\sqrt{\pi}}[w^{-\frac{3}{2}}+w^{-\frac{1}{2}}+2w^{\frac{1}{2}}], \qquad (16)$

Applying the inverse sawi transform to each side of (16), results in

$$\varphi(x)=\frac{1}{\sqrt{\pi}}\{S^{-1}[w^{-\frac{3}{2}}]+S^{-1}[w^{-\frac{1}{2}}]+2S^{-1}[w^{\frac{1}{2}}]\},$$

Then $\varphi(x)=\frac{1}{\pi}[x^{-\frac{1}{2}}+2x^{\frac{1}{2}}+\frac{8}{3}x^{\frac{3}{2}}], \qquad (17)$

Which is the required solution of (14).

Example 4.3

Consider the Abel's integral equation

$$3x^2=\int_0^x \frac{1}{\sqrt{x-t}}\varphi(t)dt, \qquad (18)$$

When we take the Sawi transform of (18), we get

$$3S[x^2]=S[\int_0^x \frac{1}{\sqrt{x-t}}\varphi(t)dt],$$

$$\Rightarrow 6w = S[x^{-\frac{1}{2}} * \varphi(x)], \qquad (19)$$

Using the Sawi transform's convolution theorem in (19), we obtain

$$6w = w^2 S[x^{-\frac{1}{2}}] S[\varphi(x)],$$

$$6w = w^2(\sqrt{\pi}\, \mathrm{w}^{-\frac{3}{2}}) S[\varphi(x)],$$

$$\Rightarrow S[\varphi(x)] = \frac{6}{\sqrt{\pi}} w^{\frac{1}{2}}, \qquad (20)$$

Applying the inverse sawi transform to each side of (20), results in

$$\varphi(x) = \frac{6}{\sqrt{\pi}} S^{-1}[w^{\frac{1}{2}}],$$

Then $\varphi(x) = \dfrac{8x^{\frac{3}{2}}}{\pi}$, ..(21)

This is a required solution to .(18).

Example 4.4

Solve the integral equation of Abel

$$\pi(\mathrm{x}+1) = \int_0^x \frac{1}{\sqrt{x-t}} \varphi(t)dt, \qquad (22) .$$

When we take the Sawi transform of (22), we get

$$S[\pi(x+1)] = S[\int_0^x \frac{1}{\sqrt{x-t}} \varphi(t)dt],$$

$$\Rightarrow \pi(S[x] + S[1]) = S[x^{-\frac{1}{2}} * \varphi(x)],$$

$$\Rightarrow \pi(1 + 1/w) = S[x^{-\frac{1}{2}} * \varphi(x)], \qquad (23)$$

Using the Sawi transform's convolution theorem in (23), we obtain

$$\pi(1 + 1/w) = w^2 S[x^{-\frac{1}{2}}] S[\varphi(x)],$$

Then $\pi(1 + 1/w) = w^2(\sqrt{\pi} w^{-\frac{3}{2}}) S[\varphi(x)]$,

$$\Rightarrow S[\varphi(x)] = \sqrt{\pi}(w^{-\frac{1}{2}} + w^{-\frac{3}{2}}), \qquad (24)$$

Applying Inverse Sawi transform to both sides of (24) we get:

$$\varphi(x) = \sqrt{\pi}(\mathrm{S}^{-1}[\mathrm{w}^{-\frac{1}{2}}] + \mathrm{S}^{-1}[\mathrm{w}^{-\frac{3}{2}}]),$$

$$\Rightarrow \varphi(x) = 2\sqrt{x} + \frac{1}{\sqrt{x}}, \qquad .(25)$$

This is a required solution to .(22).

5. Conclusion:

We successfully utilized the Sawi transformation in this work to solve the first class of Abel's integral equations. The findings of numerical examples indicated that the exact solution was obtained with minimal computational time and work. The proposed method, can be utilized to solve more singular integral equations, and associated systems.

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